Math 171A Practice Midterm I

Notes: 1) For computational problems, no credit will be given for unsupported answers gotten directly from a calculator. 2) For proof problems, no credit will be given for wrong reasons.

1. Consider the feasible set $F$ defined by the following constraints

$$x_1 - x_2 + 1 \geq 0, \quad x_2 - x_1 + 1 \geq 0, \quad x_1 - 2x_2 + 1 \geq 0, \quad x_2 - 2x_1 + 1 \geq 0.$$ 

Determine whether the linear function $2x_1 + 3x_2$ is bounded or unbounded from above. If bounded, find a maximizer; if unbounded, explain why.

2. Consider the linear system $Ax = b$ where

$$A = \begin{bmatrix} 2 & 2 & -2 & -2 \\ 3 & -3 & 3 & -3 \end{bmatrix}, \quad b = \begin{bmatrix} 4 \\ -4 \end{bmatrix}.$$ 

Find all basic solutions to this linear system.

3. Consider the ELP: minimize $c^T x$ subject to $Ax = b$, where $A, b, c$ are given as

$$A = \begin{bmatrix} 1 & 2 & 3 & -4 \\ 1 & 2 & -3 & 4 \end{bmatrix}, \quad b = \begin{bmatrix} 5 \\ -1 \end{bmatrix}, \quad c^T = [1 \ 2 \ 9 \ -12].$$

Determine whether this ELP has a minimizer or not. If yes, find a minimizer and the minimum value; if no, explain why.

4. Let $A \in \mathbb{R}^{m \times n}, c \in \mathbb{R}^n, b \in \mathbb{R}^m$. Suppose the ELP

$$\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad Ax = b
\end{align*}$$

has a minimizer $x^*$ and $c^T x^* = 1$. Show that $c^T x = 1$ for every $x$ from the feasible set.
Solutions

1. Draw the feasible set in the plane and the level curves of the objective. It can be seen that the function is bounded from above, and the maximizer is (1, 1).

2. Choose all possible two columns $a_i, a_j$, with $i < j$, and then solve the 2-by-2 square system:

$$[a_i \ a_j]x = b.$$  

There are totally four basic solutions

$$(1/3, 5/3, 0, 0), (1/3, 0, -5/3, 0), (0, 5/3, 0, -1/3), (0, 0, -5/3, -1/3).$$

3. This ELP has a minimizer, because $c = A^T \lambda$ with

$$\lambda = (2, -1).$$

The linear system $Ax = b$ is compatible. The basic solution $(0, 1, 1, 0)$ is a minimizer. The minimum value is 11.

4. Since the ELP has a minimizer, there exists $\lambda$ such that $c = A^T \lambda$. So

$$\lambda^T b = \lambda^T Ax^* = (A^T \lambda)^T x^* = c^T x^* = 1.$$

For every feasible $x$, we can get

$$c^T x = (A^T \lambda)^T x = \lambda^T Ax = \lambda^T b = 1.$$  

Thus, $c^T x = 1$ for all feasible $x$. 

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