1. (10 points) Find the minimizer of the following LP:

\[
\begin{align*}
\text{minimize} & \quad 7x_1 - 9x_2 \\
\text{subject to} & \quad 3x_1 + 4x_2 \leq 5, \quad -5x_1 + 3x_2 \leq 4, \quad 4x_1 - 5x_2 \leq 3, \quad -4x_1 - 3x_2 \leq 5.
\end{align*}
\]

Do this by drawing the feasible set and check corner points.

The corners of the feasible region plotted above with their objective values are:

- \((\frac{37}{31}, \frac{11}{31}) : \frac{160}{31}\)
- \((-\frac{1}{29}, \frac{37}{29}) : -\frac{340}{29}\)
- \((-1, -\frac{1}{3}) : -4\)
- \((-1, -\frac{1}{2}) : -\frac{5}{2}\)

Which gives a minimum of \(-\frac{340}{29}\) at the point \((-\frac{1}{29}, \frac{37}{29})\).

2. (10 points) Consider the following LP:

\[
\begin{align*}
\text{minimize} & \quad 2x_1 + 3x_2 + 4x_3, \\
\text{subject to} & \quad 3x_1 - 4x_2 - 5x_3 \geq 6, \\
& \quad x_1 + x_2 + x_3 = 10, \\
& \quad x_1 \geq 0, x_2 \geq 0, x_3 \geq 0.
\end{align*}
\]

Eliminate the equality constraint by replacing \(x_3\) in terms of \(x_1, x_2\), and convert it into an equivalent LP with only inequality constraints. Then find the minimizer \((x_1^*, x_2^*, x_3^*)\) for the above LP, by drawing pictures.
Solution: Substituting $x_3 = 10 - x_1 - x_2$, we obtain

\[
\begin{align*}
\text{minimize} & \quad -2x_1 - x_2 + 40, \\
\text{subject to} & \quad 8x_1 + x_2 \geq 56, \\
& \quad x_1 \geq 0, x_2 \geq 0, \\
& \quad x_1 + x_2 \leq 10.
\end{align*}
\]

The feasible region is the enclosed triangle, and we have plotted the isolines of the objective function in gray. The objective function is decreasing as we move to the right. So we observe that a minimum is reached at the point $(10, 0)$. The minimum for the original problem is then at $(10, 0, 0)$ with a value of 20.

3. (10 points) Find the maximizer of the following LP:

\[
\begin{align*}
\text{maximize} & \quad 3x_1 + 5x_2 + 7x_3 \\
\text{subject to} & \quad 2 \leq x_1 + x_2 \leq 3 \\
& \quad 4 \leq x_2 + x_3 \leq 5 \\
& \quad x_1 \geq 0, x_2 \geq 0, x_3 \geq 0
\end{align*}
\]

by using the Matlab function `linprog`.

Solution:

\[
\text{ans} = [3; \ 0; \ 5]
\]

The maximum value is then $f^* [3; 0; 5] = 44$.

4. (10 points) Convert the following optimization problem

\[
\begin{align*}
\text{minimize} & \quad 3|x| + 4|y| \\
\text{subject to} & \quad 5x + 7y \leq -1
\end{align*}
\]

equivalently into an LP form. Find the minimizer by using Matlab function `linprog`.

Solution: Introduce four new variables $x^+, x^-, y^+, y^-$ and convert into:
minimize \[ 3x^+ + 3x^- + 4y^+ + 4y^- \]

subject to
\[
\begin{align*}
5x + 7y & \leq -1 \\
x &= x^+ - x^- \\
y &= y^+ - y^- \\
x^+ &\geq 0 \\
x^- &\geq 0 \\
y^+ &\geq 0 \\
y^- &\geq 0
\end{align*}
\]

Now we remove the equality constraints by writing \( x \) and \( y \) in terms of \( x^+, x^-, y^+, \) and \( y^- \).

\[
\begin{align*}
\minimize & \quad 3x^+ + 3x^- + 4y^+ + 4y^- \\
\subjectto & \quad 5x^+ - 5x^- + 7y^+ - 7y^- \leq -1 \\
& \quad x^+ \geq 0 \\
& \quad x^- \geq 0 \\
& \quad y^+ \geq 0 \\
& \quad y^- \geq 0
\end{align*}
\]

Our Matlab program is then:
\[
\text{ans} = [0; 0; 0; 0.1429]
\]

We interpret the solution as \( x = 0, y = -0.1429 \), where we reach a minimum value of 0.5714.

5. (10 points) Define function \( f(x, y) = \max\{2x + y, x + 2y, x + y\} \). Convert the following optimization problem
\[
\begin{align*}
\minimize & \quad f(x, y) \\
\subjectto & \quad |x| \leq y + 2 \\
& \quad y \leq 2
\end{align*}
\]

into an LP form and find the minimizer by using MATLAB function \texttt{linprog}.

Solution: It is equivalent to the following LP problem
\[
\begin{align*}
\minimize & \quad t \\
\subjectto & \quad t \geq 2x + y \\
& \quad t \geq x + 2y \\
& \quad t \geq x + y \\
& \quad x \leq y + 2 \\
& \quad x \geq -y - 2 \\
& \quad y \leq 2
\end{align*}
\]

The minimizer is \( x = 0, y = -2 \) obtained by MATLAB.

6. (10 points) Consider the following LP:
\[
\begin{align*}
\minimize & \quad y - ax \\
\subjectto & \quad x + 2y \leq 4, -y + 2x \leq 3, x \geq 0, y \geq 0
\end{align*}
\]

where \( a \) is a constant. For what value of \( a \), the solution(s) of this LP are following points respectively?

(a) \( x = 0, y = 0 \) \quad (b) \( x = 2, y = 1 \) \quad (c) \( x = \frac{3}{2}, y = 0 \)

(e) all points in \{\( (x, y)\mid 2x - y = 3, \frac{3}{2} \leq x \leq 2 \}\} \quad (f) \text{all points in} \{\( (x, 0)\mid 0 \leq x \leq \frac{3}{2} \}\}
When $a < 0$, the solution is $x = 0, y = 0$. When $0 < a < 2$, the solution is $x = \frac{3}{2}, y = 0$. When $a > 2$, the solution is $x = 2, y = 1$. When $a = 0$, the solution is $\{(x,0)|0 \leq x \leq \frac{3}{2}\}$. When $a = 2$, the solution is $\{(x,y)|2x - y = 3, \frac{3}{2} \leq x \leq 2\}$.

7. (10 points) Suppose $a_1, a_2, a_3$ are linearly independent vectors. Determine whether $a_1 + a_2, a_2 + a_3, a_3 + a_1$ are linearly independent or not.

Solution:

The most straightforward approach is probably to directly use the definition of linear independence. That is, suppose $\alpha_1 \cdot (a_1 + a_2) + \alpha_2 \cdot (a_1 + a_3) + \alpha_3 \cdot (a_2 + a_3) = 0$ for scalars $\alpha_1, \alpha_2, \alpha_3$, and show that $\alpha_1 = \alpha_2 = \alpha_3 = 0$. To that end, we obtain:

$$(\alpha_1 + \alpha_2) \cdot a_1 + (\alpha_1 + \alpha_3) \cdot a_2 + (\alpha_2 + \alpha_3) \cdot a_3 = 0$$

Since $a_1, a_2, a_3$ are linearly independent, this gives us the system:

$$\begin{align*}
\alpha_1 + \alpha_2 &= 0 \\
\alpha_1 + \alpha_3 &= 0 \\
\alpha_2 + \alpha_3 &= 0
\end{align*}$$

Which easily gives us $\alpha_1 = \alpha_2 = \alpha_3 = 0$, and an affirmative answer.