1. Consider the \( k \)th iteration of the simplex method as defined in Algorithm 4.2 of the textbook.

(a) Show that the matrix \( A_{k+1} \), defined by replacing the \( s \)th row of \( A_k \) by the \( t \)th row of \( A_k \) is nonsingular. (The index \( t \not\in \mathcal{W}_k \) is such that \( \alpha_k = \sigma_t \), and the \( t \)th constraint is called a blocking constraint.)

Proof. Let \( A_k = \begin{bmatrix} a_1 \\
\vdots \\
a_s \\
\vdots \\
a_n \end{bmatrix} \), and new matrix \( A_{k+1} = \begin{bmatrix} a_1 \\
\vdots \\
a_t \\
\vdots \\
a_n \end{bmatrix} \). As \( A_k \) is nonsingular, so rows \( a_1, \ldots, a_{s-1}, a_{s+1}, \ldots, a_n \) are linearly independent. If \( A_{k+1} \) is singular, then

\[
a_t = \sum_{i \in \{1, \ldots, s-1, s+1, \ldots, n\}} \lambda_i a_i
\]

And the choice of \( P_k \) satisfies

\[
A_k p_k = e_s, \text{ i.e. } a_s^\top p_k = 1, \ a_i^\top p_k = 0, \forall i \neq s.
\]

The choice of \( t \) with \( a_t^\top p_k < 0 \). But in \( A_{k+1} \),

\[
a_t^\top p_k = \sum_{i \in \{1, \ldots, s-1, s+1, \ldots, n\}} \lambda_i a_i^\top p_k = 0,
\]

which is a contradiction. \( \square \)

(b) Show that the component of the Lagrange multiplier \( \lambda_a \) at \( x_{k+1} \) corresponding to the new constraint in the working set must be positive. (This implies that it is impossible to delete the constraint that was just added.)

Proof. The choice of \( p_k \) satisfies \( A_k p_k = e_s \), where \( s \) is the index for \( A_k^\top \lambda = c \), with \( \lambda_s < 0 \) and \( c^\top p_k < 0 \). New \( A_{k+1}^\top \lambda_{k+1} = c \),

\[
c^\top p_k = \lambda_{k+1}^\top A_{k+1} p_k = \lambda_{k+1}^\top [0, \ldots, a_t^\top p_k, 0, \cdots, 0]^\top = \lambda_s^\top a_t^\top p_k < 0.
\]

The choice of \( t \) satisfies constraint \( a_t^\top p_k < 0 \), which implies \( \lambda_{k+1}^s > 0 \), so it is impossible to delete the constraint. \( \square \)
2. (10 points) Consider the LP of minimizing $c^T x$ subject to $Ax \geq b$ where

$$A = \begin{bmatrix}
1 & 1 & 0 \\
0 & 1 & 1 \\
1 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}, \quad b = \begin{bmatrix}
2 \\
5 \\
4 \\
1 \\
1 \\
1
\end{bmatrix}, \quad c = A^T \begin{bmatrix}
0 \\
1 \\
1 \\
1 \\
0 \\
0
\end{bmatrix}.$$ 

Find the minimizer of this LP by using optimality conditions (do not use simplex method). Also explain why your solution is a minimizer.

Solution: We first find a candidate Lagrange multiplier to satisfy the second optimality condition. Here, the form of $c$ suggests we try $\lambda = [0, 1, 1, 1, 0, 0]^T$. Then using the third optimality condition ($\lambda_i (a_i^T x - b_i) = 0$ for every $i$), we obtain the system:

$$A_{\{2,3,4\}} x = b_{\{2,3,4\}}$$

Solving yields $x^* = (1, 2, 3)^T$. It is straightforward to check that $x^*$ is feasible, thus satisfying the first optimality condition.

3. (10 points) Let $a = (a_1, \ldots, a_n) \in \mathbb{R}^n_+$ be a positive vector. Consider the LP:

Minimize $x_1 + x_2 + \cdots + x_n$

subject to $a_1 x_1 + a_2 x_2 + \cdots + a_n x_n \geq 1$

$x_1 \geq 0, \ldots, x_n \geq 0$.

Find the solution $x^*$ of the above LP by using optimality conditions (i.e., do not use the simplex method), and express it in terms of $a_i$. Explain why your solution is optimal.

Solution: First find candidate's Lagrange multiplier. The system $c = A^T \lambda$ yields the equations:

$$1 = a_i \lambda_0 + \lambda_i$$

Letting $\lambda_0$ be the free variable, we obtain the inequalities:

$$0 \leq 1 - a_i \lambda_i \Rightarrow a_i \lambda_i \leq 1$$

Which shows us that $\lambda_i \leq \min_i \left( \frac{1}{a_i} \right) = \frac{1}{\max_i a_i}$. We choose as a candidate $\lambda_0 = \frac{1}{\max_i a_i} = \frac{1}{a_s}$ (for appropriate $s$). But then the third optimality condition will yield $x = \frac{1}{a_s} e_s$. The row corresponding to $x_s \geq 0$ will become the only inactive constraint, hence $x_i = 0$ for all $i \neq s$, and $x_s$ will be determined by the top constraint ($a_1 x_1 + a_2 x_2 + \cdots + a_n x_n \geq 1$), which is active. Checking feasibility is straightforward.

4. (5 points) Find the solution of the following LP

Minimize $c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$

subject to $\ell \leq x \leq u$

and express it in terms of $c_i, \ell_i, u_i$. Here $\ell$ and $u$ are vectors whose components are all finite and satisfy $\ell_i \leq u_i$ for $i = 1 : n$. Do not use the simplex method.

Solution: We can minimize the expression $c_1 x_1 + \cdots + c_n x_n$ via picking each $x_i$ to minimize the quantity $c_i x_i$ individually. This is done by choosing $x_i = l_i$ if $c_i \geq 0$ and $x_i = u_i$ if $c_i < 0$. 

2
5. (15 points) Consider the constraints

\[ 1 \leq 2x_1 + x_2 \leq 4, \quad 0 \leq x_1 \leq 2, \quad x_2 \geq 0. \]

(a) Formulate a phase-1 LP with a single artificial variable. Define the phase-1 problem in such a way that you know an initial vertex for the phase-1 constraints in which \( x_1 = x_2 = 0. \)

Solution: Defining the following vectors and matrices,

\[
A = \begin{bmatrix} 2 & 1 \\ -2 & -1 \\ -1 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ -4 \\ -2 \end{bmatrix}
\]

The constraints can be equivalently written as

\[ Ax \geq b, \quad x \geq 0 \]

With this, we can formulate the following phase-1 LP

\[
\begin{align*}
\text{Minimize} & \quad \xi \\
\text{subject to} & \quad Ax + \xi e \geq b \\
& \quad \xi \geq 0 \\
& \quad x \geq 0
\end{align*}
\]

which we know has the vertex \( x = (0, 0, b_{max})^T = (0, 0, 1)^T. \)

(b) Solve the phase-1 problem of part (a) by using the simplex method for problems in all inequality form.

Solution: Here’s the MATLAB code, with our matrices in MATLAB encoding ALL the constraints.

\[
\begin{align*}
\text{>> } & A = [2 \ 1 \ 1; -2 \ -1 \ 1; -1 \ 0 \ 0; 1 \ 0 \ 0; 0 \ 1 \ 0; 0 \ 0 \ 1] \\
\text{>> } & b = [1 \ -4 \ -2 \ 0 \ 0 \ 0]’ \\
\text{>> } & x = [0 \ 0 \ 1]’ \\
\text{>> } & A*x-b \\
\text{ans } = \\
& 0 \\
& 5 \\
& 3 \\
& 0 \\
& 0 \\
& 1 \\
\text{>> } & a = [1 \ 4 \ 5] \\
\text{>> } & Aa = A(a,:) \\
\text{Aa } = \\
& 2 \ 1 \ 1
\end{align*}
\]
\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix}
\]

>> c = [0 0 1]' \\
>> Aa' \ c \\

ans =

\[
\begin{bmatrix}
1 \\
-2 \\
-1
\end{bmatrix}
\]

>> p = Aa \ [0 1 0]' \\

p =

\[
\begin{bmatrix}
1 \\
0 \\
-2
\end{bmatrix}
\]

>> alpha = 1/2 \\

alpha =

0.5000

>> xbar = x + alpha*p \\

xbar =

\[
\begin{bmatrix}
0.5000 \\
0 \\
0
\end{bmatrix}
\]

>> A*xbar - b \\

ans =

\[
\begin{bmatrix}
0 \\
3.0000 \\
1.5000 \\
0.5000 \\
0 \\
0
\end{bmatrix}
\]

>> a = [2 3 4] \\
>> Aa = A(a,:)
\[ \begin{align*}
Aa &= \\
&= \begin{bmatrix}
-2 & -1 & 1 \\
-1 & 0 & 1 \\
1 & 0 & 0
\end{bmatrix}
\end{align*} \]

\[ \gg \text{Aa' \ \ c} \]

\[ \text{ans} = \\
\begin{bmatrix}
0 \\
1 \\
1
\end{bmatrix} \]

So our solution to the phase-1 problem is \( \bar{x} = (0.5, 0, 0)^T \).

(d) Use your phase-1 solution to define a feasible point for the original constraints. Is your solution a vertex for phase 2? Justify your answer.

Solution: The point \( \bar{x} = (0.5, 0, 0)^T \) is clearly feasible with respect to the original constraints. Check \( A\bar{x} - b \) has at least two zeros, and the corresponding submatrix of \( A \) is linear independent.

6. (10 points) Consider the iLP: Minimize \( c^T x \) s.t. \( Ax \geq b \), where \( A = \begin{bmatrix}
a_1^T \\
\vdots \\
a_m^T
\end{bmatrix} \in \mathbb{R}^{m \times n}. \)

Suppose \( x^* \) is a feasible point, i.e., \( Ax^* \geq b \). Show that \( x^* \) is an optimizer if and only if there exists \( \lambda = \begin{bmatrix}
\lambda_1 \\
\vdots \\
\lambda_m
\end{bmatrix} \geq 0 \) such that \( A^T \lambda = c \) and \( \lambda_i(a_i^T x^* - b_i) = 0 \) for all \( i \).

Solution:

**Proof.** Assume \( x^* \) is an minimizer of this iLP. Let \( W \) be the working set of \( x^* \), then we know \( c = \sum_{i \in W} a_i \mu_i \) for some \( \mu \in \mathbb{R}^k \) and \( \mu \geq 0 \), where \( k = \#W \). Let \( \lambda_i = \mu_i \) if \( i \in W \), and equals 0 if else, then it is easy to prove \( c = \sum_{i=1}^m a_i \lambda_i \) and \( \lambda \geq 0 \), which means \( A^T \lambda = c \). Further, if \( i \notin W \), then \( \lambda_i = 0 \), which means \( \lambda_i(a_i^T x^* - b_i) = 0 \). And if \( i \in W \), then \( a_i^T x^* - b_i = 0 \), thus \( \lambda_i(a_i^T x^* - b_i) = 0 \).

Conversely, if \( A^T \lambda = c \) and \( \lambda_i(a_i^T x^* - b_i) = 0 \). Then we have \( c = \sum_{i=1}^m a_i \lambda_i \) and \( \lambda_i \geq 0 \) for any \( i = 1, \ldots, m \). Since \( \lambda_i(a_i^T x^* - b_i) = 0 \), hence if \( a_i^T x^* - b_i \neq 0 \), then \( \lambda_i = 0 \). So \( c = \sum_{i \in W} a_i \lambda_i \), and \( \lambda_i \geq 0 \) for any \( i \in W \). where \( W \) is the set of index such that \( a_i^T x^* - b_i = 0 \), i.e., the set of indeces of active constraints. Therefore we know \( x^* \) is a minimizer. \( \Box \)

7. (10 points) Let \( A \in \mathbb{R}^{m \times n}, \ e = \begin{bmatrix}
1 \\
\vdots \\
1
\end{bmatrix} \in \mathbb{R}^m \). Suppose \( y_{n+1} > 0 \) for all \( y = \begin{bmatrix}
y_1 \\
\vdots \\
y_{n+1}
\end{bmatrix} \in \mathbb{R}^m \).
satisfying $[A \ e]y \geq b$. Show that the set $\{Ax \geq b\}$ is empty.

Solution:

Proof. Assume the set $\{Ax \geq b\}$ is not empty, then $\exists x \in \mathbb{R}^n$ such that $Ax \geq b$. Thus $y = \begin{bmatrix} x_1 \\ \vdots \\ x_n \\ 0 \end{bmatrix}$ satisfies $[A \ e]y \geq b$ but here the last entry of $y$ equals 0, which is contradictory to the condition. Thus $\{Ax \geq b\}$ is empty. \qed

8. (10 points) Let

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & 3 \\ -3 & 0 & -3 \\ 4 & 2 & 5 \end{bmatrix}, b = \begin{bmatrix} 1 \\ 6 \\ -6 \\ 11 \end{bmatrix}, c = A^T \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}.$$  

Consider the iLP:

$$\begin{cases} \text{Minimize} & c^T x \\ \text{subject to} & Ax \geq b \end{cases}.$$  

Use optimality condition to determine if it has minimizer or not. If yes, please find one. If no, please give reasons.

Proof. By solving linear system $A^T \lambda = c$, we get

$$\lambda = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$  

The Lagrange multiplier $\lambda$ should be nonnegative, so $t \geq 1$. When $t = 1$,

$$\lambda = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$  

By optimality condition, $a_i^T x - b_i = 0, i = 2, 3, 4$. So the candidate minimizer $x$ is

$$x = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$  

But $a_1^T x - b_1 = -1 < 0$, so $x$ is not feasible.

When $t > 1$, we have $\lambda > 0$. So by optimality condition, $a_i^T x - b_i = 0, i = 1, 2, 3, 4$. This linear system does not have a solution.

In conclusion, this LP does not have minimizer. \qed
9. (10 points) Consider the iLP:

\[
\begin{align*}
\text{Minimize} & \quad c^T x \\
\text{subject to} & \quad Ax \geq b
\end{align*}
\]

Suppose \( u \neq v \) are two distinct minimizers.

(a) Show that \( p_1 := v - u \) is a feasible direction at \( u \) and \( p_2 := u - v \) is a feasible direction at \( v \).

(b) Show that \( c^T u = c^T v \), so the minimum value is unique.

**Proof.** (a) When \( 0 \leq \alpha \leq 1 \),

\[
A(u + \alpha p_1) = A((1 - \alpha)u + \alpha v) = (1 - \alpha)Au + \alpha Av \geq b \implies u + \alpha p_1 \text{ is feasible}
\]

\[
A(v + \alpha p_2) = A((1 - \alpha)v + \alpha u) = (1 - \alpha)Av + \alpha Au \geq b \implies v + \alpha p_2 \text{ is feasible}
\]

Thus \( p_1, p_2 \) are feasible directions at \( u, v \) respectively.

(b) Since \( u, v \) are minimizers, it holds that \( c^T p_1 \geq 0, c^T p_2 \geq 0 \).

\[
c^T u = c^T (v + p_2) = c^T v + c^T p_2 \geq c^T v
\]

\[
c^T v = c^T (u + p_1) = c^T u + c^T p_1 \geq c^T u
\]

As a result, \( c^T u = c^T v \)