Math 171A Homework Assignment #5

**Due Date:** February 23, 2018

1. (10 points) Consider the iLP:

Minimize \( 3x_1 - x_2 + 2x_3 \)
subject to
\[
\begin{align*}
  -2x_1 + 4x_2 + 4x_3 & \geq 6 \\
  x_1 + 4x_2 + x_3 & \geq 5 \\
  -2x_1 + x_2 + 2x_3 & \geq 1 \\
  2x_1 - 2x_2 & \geq 0 \\
  -3x_2 + x_3 & \geq -2 \\
  x_1 & \geq 0 \\
  x_2 & \geq 0 \\
  x_3 & \geq 0
\end{align*}
\]

and the point \( \bar{x} = (1,1,1)^T \). Find the active set at \( \bar{x} \) and determine if the point \( \bar{x} \) is optimal. If \( \bar{x} \) is not optimal, find a direction \( p \) such that \( c^T p < 0 \) and \( A_a(\bar{x})p \geq 0 \).

2. (10 points) Consider the iLP: Minimize \( c^T x \) subject to \( Ax \geq b \), where

\[
A = \begin{bmatrix}
-1 & -3 & -1 \\
1 & 0 & -3 \\
0 & 1 & -2 \\
-2 & -3 & 1 \\
1 & 5 & 1
\end{bmatrix}, \quad b = \begin{bmatrix}
-3 \\
-2 \\
-4 \\
-2 \\
-6
\end{bmatrix}, \quad c = \begin{bmatrix}
-5 \\
-5 \\
3
\end{bmatrix}.
\]

(a) Verify that \( x^{(0)} = (1,1/3,1)^T \) is a vertex. What is the active set?

(b) Perform one iteration of the simplex method (Algorithm 4.2 in the textbook) to obtain the next iterate \( x^{(1)} \). Verify that \( x^{(1)} \) is feasible and that \( c^T x^{(1)} < c^T x^{(0)} \). What is the active set at \( x^{(1)} \)? Show your work.

(c) Repeat part (b) until a minimizer is found.
3. (10 points) Consider the following five constraints
\[ x_1 + 2x_2 \leq 3, \quad x_1 - x_2 \geq 0, \quad 2x_1 + x_2 \leq 3, \quad x_1 + 5x_2 \leq 6, \quad x_1 - 2x_2 \geq -1. \]
(a) Find a degenerate vertex \( x^{(0)} \) (by sketching the feasible region).
How many possible working set matrices \( A_0 \) are there at \( x^{(0)} \)?
(b) Suppose that we wish to minimize \( x_1 + x_2 \) subject to these constraints, starting at \( x^{(0)} \) and using the simplex method. Find a working set matrix \( A_0 \) for which the Lagrange multiplier vector (the solution of \( A_0^T \lambda = c \)) contains at least one negative component \( \lambda_s \), but the simplex search direction satisfying \( A_0p = e_s \) is not a feasible descent direction (by inspecting the picture of the feasible region).
(c) Under the same conditions as in part (b), find a working set matrix \( A_0 \) for which the Lagrange multiplier vector contains at least one negative component, but the associated search direction \( p \) is a feasible descent direction (by inspecting the picture of the feasible region).

4. (10 points) Let \( A \in \mathbb{R}^{m \times n} \). Suppose \( x \geq 0 \) for all \( x \in \mathbb{R}^n \) satisfying \( Ax \geq 0 \). Show that there exists a matrix \( B \in \mathbb{R}^{m \times n} \) such that all entries of \( B \) are nonnegative and \( A^T B = I_n \).

5. (10 points) (i) Assume that \( p \in \text{Null}(A) \). If \( a \) is such that \( a^T p \neq 0 \), show that \( a \) must be linearly independent of the rows of \( A \). (ii) If \( a \) is linearly independent of the rows of \( A \) and \( p \in \text{Null}(A) \), is \( a^T p \neq 0 \)? Justify your answer.

6. (10 points) Consider the iLP : minimize \( c^T x \) subject to \( Ax \geq b \), where
\[
A = \begin{bmatrix} a_1^T \\ \vdots \\ a_m^T \end{bmatrix} \in \mathbb{R}^{m \times n}, \quad b = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}.
\]
Assume all vertices are nondegenerate, \( x_0 \) is a vertex whose working set is \( \mathcal{W}_0 \), \( A_0 \) denote the matrix whose rows are the rows of \( A \) specified by \( \mathcal{W}_0 \). \( \lambda \) and \( p \) are vectors satisfying \( A_0 \lambda = c \) and \( A_0p = e_s \), where \( (\lambda)_s < 0 \). Let \( \mathcal{D} \) be the set of indices of decreasing constraints, \( r_j(x_0) = a_j^T x_0 - b_j \). For \( j \notin \mathcal{W}_0 \) define
\[
\sigma_j = \begin{cases} \frac{r_j(x_0)}{-a_j^T p} & \text{if } j \in \mathcal{D} \\ \infty & \text{otherwise} \end{cases}
\]
and $\alpha = \sigma_t = \min_{j \notin W_0} \sigma_j$. Show that if $\alpha < +\infty$, then $x_0 + \alpha p$ is a vertex whose working set is $W_0 - \{s\} + \{t\}$, and $c^T x_0 < c^T (x_0 + \alpha p)$. 

7. (10 points) Let $A \in \mathbb{R}^{m \times n}$, $c \in \mathbb{R}^n$, $c \neq 0$. If $Ac < 0$, show that there is no $x \in \mathbb{R}^m$ such that $A^T x = c$ and $x \geq 0$. 

8. (10 points) Let 

\[
A = \begin{bmatrix}
1 & 0 & 1 \\
1 & 0 & -2 \\
1 & 1 & -3 \\
0 & 1 & 1 \\
2 & 1 & 0
\end{bmatrix}, b = \begin{bmatrix}
2 \\
-1 \\
-1 \\
0 \\
1
\end{bmatrix}, c = \begin{bmatrix}
-1 \\
0 \\
-4
\end{bmatrix}
\]

Consider the iLP 

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad Ax \geq b
\end{align*}
\]

First check that $\bar{x} = (1, 1, 1)$ is a vertex of this optimization problem and write down the active set of $\bar{x}$. Use simplex method to solve this iLP.

9. (10 points) Consider the following iLP 

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad Ax \geq b
\end{align*}
\]

Show that if the iLP has a minimizer and the equation $Ax = b$ is compatible, then each $\bar{x}$ satisfying $A\bar{x} = b$ is an optimizer.