1. Problem 1: \( A \) is a \( 3 \times 4 \) matrix. So we should pick 3 columns from 4. Use columns \{1, 2, 3\}, we get linear system:

\[
\begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & 2 \\
1 & 2 & 3
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} =
\begin{bmatrix}
3 \\
5 \\
7
\end{bmatrix}
\]

It has solution \((x_1, x_2, x_3) = (1, 0, 2)\), so it corresponds to basic solution \((1, 0, 2, 0)^T\).

Use columns \{1, 2, 4\}, the linear system is

\[
\begin{bmatrix}
1 & 1 & 2 \\
1 & 1 & 3 \\
1 & 2 & 6
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_4
\end{bmatrix} =
\begin{bmatrix}
3 \\
5 \\
7
\end{bmatrix}
\]

It has solution \((x_1, x_2, x_4) = (3, -4, 2)\), so it corresponds to basic solution \((3, -4, 0, 2)^T\).

Use columns \{1, 3, 4\}, the linear system is

\[
\begin{bmatrix}
1 & 1 & 2 \\
1 & 2 & 3 \\
1 & 3 & 6
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_3 \\
x_4
\end{bmatrix} =
\begin{bmatrix}
3 \\
5 \\
7
\end{bmatrix}
\]

It has solution \((x_1, x_3, x_4) = (1, 2, 0)\), so it corresponds to basic solution \((1, 0, 2, 0)^T\).

Use columns \{2, 3, 4\}, the linear system is

\[
\begin{bmatrix}
1 & 1 & 2 \\
1 & 2 & 3 \\
2 & 3 & 6
\end{bmatrix}
\begin{bmatrix}
x_2 \\
x_3 \\
x_4
\end{bmatrix} =
\begin{bmatrix}
3 \\
5 \\
7
\end{bmatrix}
\]

It has solution \((x_2, x_3, x_4) = (2, 3, -1)\), so it corresponds to basic solution \((0, 2, 3, -1)^T\).

2. We can draw a level curve for this problem as follow,

and from the level curve we know the optimizer is \(x_1 = 3, x_2 = 0\), minimum value is 3.
3. The feasible region is The direction vectors of the boundary are \( p_1 = (-2, 1), p_2 = (-4, 1) \). So \( c = (1, t)^T \) should satisfy \( \{ c^T p_1 = -2 + t > 0, c^T p_2 = -4 + t < 0 \} \) or \( \{ c^T p_1 = -2 + t < 0, c^T p_2 = -4 + t > 0 \} \). By solving these inequalities, we get \( 2 < t < 4 \). \( \bar{x} = (0, 0) \in F \), so \( \bar{x} + \alpha p_1 \in F, \bar{x} + \alpha p_2 \in F \) for all \( \alpha \geq 0 \). When \( 2 < t < 4 \),

\[
c^T (\bar{x} + \alpha p_1) = \alpha (-2 + t) \to +\infty, \text{ as } \alpha \to +\infty
\]

\[
c^T (\bar{x} + \alpha p_2) = \alpha (-4 + t) \to -\infty, \text{ as } \alpha \to +\infty
\]

which means \( c^T x = x_1 + tx_2 \) is unbounded from both below and above when \( 2 < t < 4 \).

4. It is easy to check this ELP is feasible. Let \( A \in \mathbb{R}^{2 \times 3} \) and \( b \in \mathbb{R}^2 \) such that the equation
constraint is $Ax = b$, then this ELP has a minimizer if and only if

$$c := \begin{bmatrix} 2 \\ -3 \\ c_3 \end{bmatrix} \in \text{Range}(A^T),$$

which means $A^T x = c$ is compatible. This is equivalent to $c_3 = -29$. And when $c_3 = -29$, the solution of $A^T x = c$ is $\begin{bmatrix} 7 \\ -5 \end{bmatrix}$. That is, the Lagrange Multiplier of this ELP is $\lambda = \begin{bmatrix} 7 \\ -5 \end{bmatrix}$, thus the minimum value of this ELP is $\lambda^T b = 11$. 