

# Kac-Moody groups over the last decade.

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# Outline

- Kac-Moody groups, introduction.
- The geometric property.
- The homotopical property.
- What we (do not) know.

## Introduction

In an email to C.Wilkerson and me, in Feb.1996, Haynes observes...

- There exists a topological group  $K$ , with the property:

$$H^{2n}(K, \mathbb{Z}) = H^{2n+3}(K, \mathbb{Z}) = \mathbb{Z}/\mathbb{Z}f_n,$$

where  $f_n$  is the  $2n$ -th Fibonacci number given by:

$$f_{n+1} = 3f_n - f_{n-1}, \quad f_0 = 0, \quad f_1 = 1$$

- In addition, he notices that there is a compact Lie group  $H$  of  $K$ , isomorphic to  $U(2)$ , so that the homogeneous space  $K/H$  has torsion free cohomology:

$$H^{2n}(K/H, \mathbb{Z}) = \mathbb{Z} \langle \gamma_n \rangle, \quad \gamma_n \cup \gamma_m = F(m, n) \gamma_{m+n}.$$

He calls  $F(m, n)$ , the *Generalized Binomial Coefficients*:

$$F(m, n) = \frac{f_{m+n} f_{m+n-1} \cdots f_1}{f_n \cdots f_1 f_m \cdots f_1}.$$

In particular, these numbers are integers!

- We compare this with the example:  $G = LSU(2)\langle 2 \rangle$ . The group  $G$  contains a subgroup  $H \cong S^1 \times SU(2)$ , and:

$$H^{2n}(G/H, \mathbb{Z}) = \mathbb{Z}\langle \gamma_n \rangle, \quad \gamma_n \cup \gamma_m = C(m, n) \gamma_{m+n},$$

where  $G/H = \Omega SU(2)$ , so  $C(m, n)$  are the usual Binomial coefficients.

- Infact,  $K$  belongs to an infinite family of such group  $K(a)$  for integers  $a \geq 2$ , for which the integers  $f_n$  are defined as

$$f_{n+1} = a f_n - f_{n-1}, \quad f_0 = 0, \quad f_1 = 1.$$

- The group  $K(2)$  is actually equivalent to  $LSU(2)\langle 2 \rangle$ .
- The groups  $K(a)$  is obtained by taking a suitable amalgam of two copies of  $S^1 \times SU(2)$  (if  $a$  is even), or two copies of  $U(2)$  (if  $a$  is odd), over the diagonal torus:  $S^1 \times S^1$ .
- These groups belong to the class of groups known as Kac-Moody groups. As the example illustrates, they are constructed as suitable amalgams of compact Lie groups over maximal rank embeddings.
- These groups have complexifications  $K(\mathbb{C})$ . Versions of  $K(\mathbb{C})$  can be defined over any field! The groups  $K(\mathbb{F})$  over various fields  $\mathbb{F}$  are a very rich class of new interesting groups generalizing Chevalley groups.

- Given a simple, simply connected compact Lie group  $G$ , there is a Kac-Moody group  $K$  homotopy equivalent to the central extension  $\tilde{L}G$  of the Loop group of  $G$ . Here  $\tilde{L}G$  is equivalent to the 2-connected cover  $LG\langle 2 \rangle$  of  $LG$ .
- The exceptional compact Lie group  $E_6$  fits in a family  $E_n$   $n \geq 6$ , with  $E_9 = \tilde{L}E_8$ . The group  $E_{10}(\mathbb{R})$  is conjectured to have relevance in M-theory.
- There is a notion of a Lie algebra of a Kac-Moody group. The exponential map is defined on a dense subset, and the image generates the Kac-Moody group.

Over the last ten years, many topologists have thought about these groups: J.Aguade, C.Broto, J.M.Cantarero, B.Dwyer, H.Miller, A.Ruiz, L.Saumell....among others.

## The geometric Property (Kac-Peterson, Tits):

Let  $\mathcal{C}$  be a category of subsets of the set:  $\underline{n} = \{1, 2, \dots, n\}$ , which satisfies two properties:

- (a) The singletons  $\{i\}$  belong to  $\mathcal{C}$ , for all  $i \leq n$ .
- (b) If  $I \in \mathcal{C}$ , then  $J \in \mathcal{C}$  for  $J \subseteq I$ .

• For a Kac-Moody group  $K$ , there is an  $n$ , and a  $\mathcal{C}$  such that

- (1)  $K$  is an amalgam (in the category of topological groups), of compact connected Lie groups  $H_I$ , for  $I \in \mathcal{C}$ .
- (2)  $H_\emptyset$  is the common maximal torus  $T \subset H_I$ ,  $\forall I \in \mathcal{C}$ .
- (3)  $H_I$  has semi-simple rank  $|I|$ , and is generated by  $H_{\{i\}}$ , for  $i \in I$ .

## Some consequences of the Geometric Property:

(a) The normalizer  $N(T)$  of  $T$  in  $K$  is also an amalgam of the normalizers  $N_I(T)$  of  $T$  in  $H_I$ . The same goes for the Weyl group  $W = N(T)/T$ , (which is a reflection group).

Examples of  $W$  include  $D_\infty$ , and  $PGL_2(\mathbb{Z})$ .

(b) The homogeneous spaces  $K/H_I$  have the structure of a  $CW$ -complex with cells only in even degrees, indexed on the cosets  $W/W_I$ , where  $W_I = N_I(T)/T$ .

(c) The representation theory of  $K$  can be completely described using the representations of  $H_I$ , and the combinatorial data involved in the gluing. Under a suitable hypothesis, representations may be detected on  $T$ , and there is a character theory for these representations.

## The Homotopical Property (K., 1998):

The next crucial property of Kac-Moody groups is the property that the following canonical map is a homotopy equivalence:

$$\operatorname{hocolim}_{\mathcal{C}} BH_I \longrightarrow BK.$$

- In fact, something stronger is true. The Tits building:

$$X := \operatorname{hocolim}_{\mathcal{C}} K/H_I,$$

is  $K$ -equivariantly contractible. The property above follows from the contractibility of  $X$  by taking homotopy orbits with respect to  $K$ .

- So for example,  $LBSU(2)\langle 3 \rangle$  is equivalent to the pushout of two copies of  $BS^1 \times BSU(2)$  over  $BS^1 \times BS^1$ .

## Some consequences of the Homotopical Prop.:

(1) A similar homotopy decomposition holds for  $BN(T)$  and  $BW$ , in terms of  $BN_I(T)$  and  $BW_I$  resp.

(2) The map

$$BN(T) \longrightarrow BK,$$

is an equivalence for all but a finite set of (bad) primes.

(3) This set of *bad primes* are those that appear as torsion of  $W$ . Since  $W \subset GL_r(\mathbb{Z})$ , for some  $r$ , this set is finite.

(4) The cohomology of the Weyl group  $W$  is trivial away from the set of bad primes.

(5) (Broto-K., 2002.) The cohomology ring  $H^*(BK, \mathbb{F}_p)$  is a finitely generated  $\mathbb{F}_p$ -algebra. Its Krüll dimension is the rank of the largest elementary abelian  $p$ -gp in  $K$ .

(6) (Broto-K., 2002) Given a finite  $p$ -group  $\pi$ , there is an equivalence:

$$\coprod_{\varphi \in \text{Rep}(\pi, K)} BC(\varphi)_{\hat{p}} \simeq \text{Map}(B\pi, BK_{\hat{p}}).$$

However, these spaces need not be  $p$ -good! (Dwyer). An algebraic version of the above, using the Lannes T-functor also holds.

- Other homotopy decompositions like the Centralizer decomposition of  $BK_{\hat{p}}$ , also hold, but may fail to be perfect.

(7) (Broto-K., 2002) Let  $\mathcal{A}$  denote the category of elementary abelian subgroups of  $K$ . Then the Quillen map:

$$H^*(BK, \mathbb{F}_p) \longrightarrow \lim_{E \in \mathcal{A}} H^*(BE, \mathbb{F}_p),$$

has a nilpotent kernel and cokernel.

## What we do (not) know:

### (I) $W$ -invariants:

Consider the map  $\varphi : K/T \longrightarrow BT$ . In cohomology with coefficients in a field  $\mathbb{F}$ , Kac shows that the kernel of  $\varphi^*$  is generated by a regular sequence:

$$I = \langle \sigma_1, \dots, \sigma_r \rangle,$$

where  $r \leq rk(T)$ , and  $r = rk(T)$  if  $\text{char}(\mathbb{F}) > 0$ . The ideal  $I$  is called *The ideal of generalized invariants* of  $K$ . It contains the Weyl-group invariants ( $\text{char}(\mathbb{F}) \neq 2$ ).

- The ideal  $I$  can be defined algebraically in terms of the action of an algebra called *Nil-Hecke algebra* on  $H^*(BT, \mathbb{F})$ . The Nil-Hecke algebra is a deformation of the group algebra of  $W$ .

- Kac shows that If  $\text{char}(\mathbb{F}) = p$ , then  $I$  agrees with the ideal generated by the  $W$ -invariants if and only if

$$|W \bmod p| = \prod_i |\sigma_i|, \quad (1)$$

where  $|\sigma_i| = \frac{1}{2} \deg(\sigma_i)$ , and  $|W \bmod p|$  denotes the order of the image of  $W$  in  $GL_r(\mathbb{F})$ .

- The above condition does not require  $|W \bmod p|$  to be prime to  $p$ . If that does hold, then by a theorem of Chevalley-Shepard-Todd, we can show a stronger statement:

$$H^*(BT, \mathbb{F}_p)^W = \mathbb{F}_p[\sigma_1, \dots, \sigma_r].$$

- **Question:** Does Kac's condition (1) hold away from the bad primes? How is this related to the image of  $H^*(BK, \mathbb{F})$  in  $H^*(BT, \mathbb{F})^W$ ?

## (II) The homology ring $H_*(K, \mathbb{F}_p)$ :

Next, consider the fibration  $K \longrightarrow K/T \longrightarrow BT$ . Let  $S$  denote the image of  $H^*(BT, \mathbb{F})$  in  $H^*(K/T, \mathbb{F})$ .

- Kac-Peterson show that  $H^*(K/T, \mathbb{F})$  is a free  $S$ -module.

- Their work shows that the Eilenberg-Moore spectral sequence collapses to give an extension of *Hopf Algebras*:

$$1 \longrightarrow H^*(K/T; \mathbb{F}) \otimes_S \mathbb{F} \longrightarrow H^*(K, \mathbb{F}) \longrightarrow \Lambda(y_1, \dots, y_r) \longrightarrow 1,$$

where  $y_i$  are odd classes in degree  $\deg(\sigma_i) - 1$ .

- The even Hopf-algebra  $\Gamma_p := H^*(K/T, \mathbb{F}_p) \otimes_S \mathbb{F}_p$  is locally finite over the Steenrod algebra and so it is not finitely generated.

- The dual:  $\Gamma_p^*$ , is a finitely generated algebra. In particular  $H_*(K, \mathbb{F}_p)$  is finitely generated.
- Recall that  $H^*(K/T, \mathbb{F})$  has a basis:  $\{\delta_w\}$ , with  $w \in W$ .

There is a coproduct on  $H^*(K/T, \mathbb{F})$  (due to D.Peterson), that induces the coproduct on  $\Gamma = H^*(K/T, \mathbb{F}) \otimes_{\mathcal{S}} \mathbb{F}$ :

$$\Delta(\delta_w) = \sum_{u*v=w} \delta_u \otimes \delta_v,$$

where  $u * v$  is defined to be  $uv$  if the lengths add, and is not defined otherwise.

- Since  $W$  may be highly non-commutative, the above formula suggests that  $\Gamma^*$  is rarely commutative.

## The Example of $K(a)$ (K., 1998):

Let  $K = K(a)$ . Then the Weyl group  $W$  is the free product of two reflections

$$W = \langle r_1, r_2 \mid r_1^2 = r_2^2 = 1. \rangle$$

The extension describing  $N(T)$  depends on  $a$  however. Notice that the only bad prime is  $p = 2$ .

- Let  $k$  be the smallest nonzero integer so that  $p$  divides an element of the sequence  $f_k$ . Notice that if  $a = 2$ , then  $k = p$ . The generalized invariants over  $\mathbb{F}_p$  are generated in degrees 4 and  $2k$ .
- $H^*(BK, \mathbb{F}_p) = \mathbb{F}_p[x_4, x_{2k}] \otimes E(x_{2k+1})$ . There is a non-trivial higher Bockstein with target  $x_{2k+1}$ . The polynomial generators detect the generalized invariants in  $H^*(BT, \mathbb{F}_p)$ .

- The  $W$  invariants agree with the generalized invariants for odd primes. For the prime 2, this may not be the case. For example, if  $a$  is even, then the action of  $W$  on  $H^*(BT, \mathbb{F}_2)$  is trivial.

- One may calculate the Hopf algebra  $H_*(K(a), \mathbb{F}_p)$ . There is an extension of Hopf algebras:

$$1 \rightarrow E(x_3, x_{2k-1}) \rightarrow H_*(K(a), \mathbb{F}_p) \rightarrow \mathbb{F}_p[x_{2k}] \rightarrow 1.$$

This splits as Hopf algebras for odd primes  $p$ .

- Rationally, one has isomorphisms:

$$H^*(K(a)/T, \mathbb{Q}) = \mathbb{Q}[x, y]/\langle x^2 + y^2 - a xy \rangle,$$

$$H^*(K(a), \mathbb{Q}) = E(x_3).$$

### (III) Other $W$ -modules:

In general, one wants to study the cohomology groups:

$$H^*(W, H^*(K/T, \mathbb{F}_p)).$$

- The  $W$ -invariants on  $H^*(K/T, \mathbb{R})$  are known for any ring  $\mathbb{R}$ , where 2 is not a zero divisor:

$$H^*(K/T, \mathbb{R})^W = \mathbb{R}.$$

- If  $p$  is not a bad prime, then:

$$H^i(W, H^*(K/T, \mathbb{F}_p)) = 0, \quad i > 0.$$

The fact that  $K/N(T)$  is acyclic away from the bad primes, also follows from the above statement.

## (IV) The Stable Transfer:

Consider the map:  $BN(T) \longrightarrow BK$ . This map is an equivalence away from the bad primes. At bad primes however, the fiber  $K/N(T)$  is not  $\mathbb{F}_p$ -finite. There is still reason to conjecture:

- **Conjecture:**  $BK$  is a stable retract of  $BN(T)$ .

This conjecture is true for the groups  $K(a)$ .

Note that each  $BH_I$  is a retract of  $BN_I(T)$ . Now  $BK$  and  $BN(T)$  are homotopy colimits of the spaces  $BH_I$  and  $BN_I(T)$  resp., so it is reasonable to attempt to construct a compatible family of transfers. Up to homotopy, this is true. The higher obstructions to doing this sometimes vanish for trivial reasons as in the case of the groups  $K(a)$ .

## (V) Homotopy Uniqueness:

It is known from the work of D. Notbohm that two compact Lie groups  $G, H$  are isomorphic if and only if  $BG$  and  $BH$  are homotopy equivalent. This statement fails for Kac-Moody groups.

J. Aguade and A. Ruiz have shown that there exist distinct Kac-Moody groups  $K(a, b)$ , such that the spaces  $BK(a, b)$  are homotopy equivalent. The family  $\{(a, b)\}$  for which this happens is finite. This leads us to the

- **Question:** Given a Kac-Moody group  $K$ , is the set of isomorphism classes of Kac-Moody groups  $K_\alpha$  for which  $BK_\alpha \simeq BK$  a finite set? What about at a prime  $p$ ? Or even better, can we define  $p$ -compact Kac-Moody groups?

## (VI) Representation theory:

Recall that the representation theory of  $K$  is a combinatorial problem in gluing the representations of  $H_I$  together. These representations tend to be infinite dimensional, and their characters (if defined) are interesting  $W$ -invariant formal expressions.

- There is a  $W$ -invariant cone in  $Lie(T)^*$ , called the Tits cone, where this problem has been solved by Kac. More precisely, one can classify all reasonable  $K$  representations whose characters belong to this cone. These are called the *Highest weight representations of  $K$* , or the positive energy representations (in the case of the Loop group).
- **Question:** Are there other families of representations?

## An example of the Denominator formula:

For the Kac-Moody group  $K(2)$ , there exist two distinguished characters  $u, v$  (given by the two simple roots).

- The formula of Kac on the character of the *trivial* Highest weight representation reduces to the Jacobi Triple-product identity:

$$\frac{\sum_{m \in \mathbb{Z}} (-1)^m u^{m(m-1)/2} v^{m(m+1)/2}}{\prod_{n=1}^{\infty} (1 - u^n v^n)(1 - u^n v^{n-1})(1 - u^{n-1} v^n)} = 1$$

- The left hand side is indeed invariant under the Weyl group  $W = \langle r_1, r_2 \mid r_1^2 = r_2^2 = 1 \rangle$ , where the action of  $r_1$  and  $r_2$  is given by:

$$r_1(u) = u^{-1}, r_1(v) = v u^2, \quad r_2(u) = u v^2, r_2(v) = v^{-1}.$$

## (VII) Dominant K-theory (K., 2008):

If  $K$  satisfies a property called *symmetrizable*, then each Highest weight representation is unitary. Consider the universal Hilbert space  $\mathcal{H}$  containing each irreducible Highest weight representation infinitely often. Let  $F(\mathcal{H})$  denote the space of Fredholm operators on  $\mathcal{H}$  in a suitable topology. Then  $K$  acts continuously on  $F(\mathcal{H})$ .

- One defines *Dominant K-theory* for space  $X$  with a proper  $K$ -action by:

$$\mathbb{K}_K^0(X) = [X, F(\mathcal{H})]_K.$$

- For the universal space  $\underline{EK}$  of proper  $K$ -actions, we can calculate  $\mathbb{K}_K^*(X)$ , to get:

- **Theorem:** Let  $K$  be such that the category  $\mathcal{C}$  consists of *all proper subsets* of  $\underline{n}$ , then:

$$\tilde{\mathbb{K}}_K^{n-1}(\underline{EK}) = \mathbb{Z}\langle HW R \rangle.$$

where the right hand side denotes the free abelian group on the set of all irreducible Highest weight representations.

- For central extensions of Loop groups  $\tilde{L}G$ , the space  $\underline{EK}$  is the affine space of principal  $G$ -connections over  $S^1$ . In this case, the above theorem follows from a theorem of Freed-Hopkins-Teleman.

- The geometric description of the above isomorphism given by FHT (using a natural equivariant family of Dirac operators) in the case of a Loop group, also holds in the above theorem.

- For any compact subgroup  $H < K$ , taking  $X = K/H$ , one gets the Dominant representation ring:

$$DR(H) := \mathbb{K}_K^0(K/H) \subseteq R(H),$$

where  $DR(H)$  can be identified with the subring of the representation ring  $R(H)$ , generated by those  $H$ -representations that appear in  $\mathcal{H}$ .

- There is a one dimensional rep.  $L \in DR(H)$ , so that:

$$R(H) = DR(H)[L^{-1}].$$

In particular, if  $H$  is a connected, semi-simple compact Lie group, then  $DR(H) = R(H)$ .

- For arbitrary compact  $H < K$ , it follows that the completion of  $DR(H)$  and  $R(H)$  at the augmentation ideal are isomorphic. This gives us:

- **Theorem:** The natural map:

$$\mathbb{K}_K^*(X) \longrightarrow K^*(EK \times_K X),$$

is given by completion with respect to the augmentation ideal on single (proper) orbits.

- Notice however, that the augmentation ideal is not well defined globally since the Highest weight representations are generally infinite dimensional.

- For finite  $K$ -CW complexes  $X$ , I believe that there is a natural global topology on  $\mathbb{K}_K^*(X)$  so that the above map is given by completion. In other words, the Atiyah-Segal completion theorem holds.

The above applies to the spaces  $\underline{EK}$ , which are finite  $K$ -CW complexes, and so we get a computation of  $K^*(BK)$ .

### (VIII) A rank 3 example:

There is a Kac-Moody group  $K$ , with  $n = 3$  and with the semisimple parts  $L_I \subset H_I$  are:

$$L_{\{1,2\}} = SU(3), \quad L_{\{1,3\}} = SU(2) \times SU(2), \quad L_{\{1\}} = SU(2),$$

where  $L_{\{1\}}$  includes as the first factor in  $L_{\{1,3\}}$ , and by the standard inclusion in  $L_{\{1,2\}}$ .

- The Weyl group of the above Kac-Moody group is isomorphic to  $PGL_2(\mathbb{Z})$ , which can therefore be written as an amalgam of  $\Sigma_3$  and  $\mathbb{Z}/2 \times \mathbb{Z}/2$  over  $\mathbb{Z}/2$ .
- The group  $K(2)$  is a (non-compact) subgroup of  $K$ , generated by  $L_{\{2\}}$  and  $L_{\{3\}}$ . Hence, the group  $D_\infty$  sits naturally inside  $PGL_2(\mathbb{Z})$ , generated by those two reflections.

## Open Questions:

- The structure of the generalized invariants has hardly been studied.
- The image of  $H^*(BK, \mathbb{F}_p) \subseteq H^*(BT, \mathbb{F}_p)^W$  is not known.
- The question of homotopy uniqueness of  $BK$  and its relation to  $BN(T)$  is completely open.
- The stable homotopy type of  $BK$  is not understood.
- Can one define “p-compact” versions of Kac-Moody groups?  
We will need to study (non-finite) p-adic pseudo-reflection groups. This was done for groups generated by two reflections by Aguade-Broto-Saumell.