

Kac-Moody groups over the last decade.

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Outline

- Kac-Moody groups, introduction.
- The geometric property.
- The homotopical property.
- What we (do not) know.

Introduction

In an email to C.Wilkerson and me, in Feb.1996, Haynes observes...

- There exists a topological group K , with the property:

$$H^{2n}(K, \mathbb{Z}) = H^{2n+3}(K, \mathbb{Z}) = \mathbb{Z}/\mathbb{Z}f_n,$$

where f_n is the $2n$ -th Fibonacci number given by:

$$f_{n+1} = 3f_n - f_{n-1}, \quad f_0 = 0, \quad f_1 = 1$$

- In addition, he notices that there is a compact Lie group H of K , isomorphic to $U(2)$, so that the homogeneous space K/H has torsion free cohomology:

$$H^{2n}(K/H, \mathbb{Z}) = \mathbb{Z} \langle \gamma_n \rangle, \quad \gamma_n \cup \gamma_m = F(m, n) \gamma_{m+n}.$$

He calls $F(m, n)$, the *Generalized Binomial Coefficients*:

$$F(m, n) = \frac{f_{m+n} f_{m+n-1} \cdots f_1}{f_n \cdots f_1 f_m \cdots f_1}.$$

In particular, these numbers are integers!

- We compare this with the example: $G = LSU(2)\langle 2 \rangle$. The group G contains a subgroup $H \cong S^1 \times SU(2)$, and:

$$H^{2n}(G/H, \mathbb{Z}) = \mathbb{Z}\langle \gamma_n \rangle, \quad \gamma_n \cup \gamma_m = C(m, n) \gamma_{m+n},$$

where $G/H = \Omega SU(2)$, so $C(m, n)$ are the usual Binomial coefficients.

- Infact, K belongs to an infinite family of such group $K(a)$ for integers $a \geq 2$, for which the integers f_n are defined as

$$f_{n+1} = a f_n - f_{n-1}, \quad f_0 = 0, \quad f_1 = 1.$$

- The group $K(2)$ is actually equivalent to $LSU(2)\langle 2 \rangle$.
- The groups $K(a)$ is obtained by taking a suitable amalgam of two copies of $S^1 \times SU(2)$ (if a is even), or two copies of $U(2)$ (if a is odd), over the diagonal torus: $S^1 \times S^1$.
- These groups belong to the class of groups known as Kac-Moody groups. As the example illustrates, they are constructed as suitable amalgams of compact Lie groups over maximal rank embeddings.
- These groups have complexifications $K(\mathbb{C})$. Versions of $K(\mathbb{C})$ can be defined over any field! The groups $K(\mathbb{F})$ over various fields \mathbb{F} are a very rich class of new interesting groups generalizing Chevalley groups.

- Given a simple, simply connected compact Lie group G , there is a Kac-Moody group K homotopy equivalent to the central extension $\tilde{L}G$ of the Loop group of G . Here $\tilde{L}G$ is equivalent to the 2-connected cover $LG\langle 2 \rangle$ of LG .
- The exceptional compact Lie group E_6 fits in a family E_n $n \geq 6$, with $E_9 = \tilde{L}E_8$. The group $E_{10}(\mathbb{R})$ is conjectured to have relevance in M-theory.
- There is a notion of a Lie algebra of a Kac-Moody group. The exponential map is defined on a dense subset, and the image generates the Kac-Moody group.

Over the last ten years, many topologists have thought about these groups: J.Aguade, C.Broto, J.M.Cantarero, B.Dwyer, H.Miller, A.Ruiz, L.Saumell....among others.

The geometric Property (Kac-Peterson, Tits):

Let \mathcal{C} be a category of subsets of the set: $\underline{n} = \{1, 2, \dots, n\}$, which satisfies two properties:

- (a) The singletons $\{i\}$ belong to \mathcal{C} , for all $i \leq n$.
- (b) If $I \in \mathcal{C}$, then $J \in \mathcal{C}$ for $J \subseteq I$.

• For a Kac-Moody group K , there is an n , and a \mathcal{C} such that

- (1) K is an amalgam (in the category of topological groups), of compact connected Lie groups H_I , for $I \in \mathcal{C}$.
- (2) H_\emptyset is the common maximal torus $T \subset H_I$, $\forall I \in \mathcal{C}$.
- (3) H_I has semi-simple rank $|I|$, and is generated by $H_{\{i\}}$, for $i \in I$.

Some consequences of the Geometric Property:

(a) The normalizer $N(T)$ of T in K is also an amalgam of the normalizers $N_I(T)$ of T in H_I . The same goes for the Weyl group $W = N(T)/T$, (which is a reflection group).

Examples of W include D_∞ , and $PGL_2(\mathbb{Z})$.

(b) The homogeneous spaces K/H_I have the structure of a CW -complex with cells only in even degrees, indexed on the cosets W/W_I , where $W_I = N_I(T)/T$.

(c) The representation theory of K can be completely described using the representations of H_I , and the combinatorial data involved in the gluing. Under a suitable hypothesis, representations may be detected on T , and there is a character theory for these representations.

The Homotopical Property (K., 1998):

The next crucial property of Kac-Moody groups is the property that the following canonical map is a homotopy equivalence:

$$\operatorname{hocolim}_{\mathcal{C}} BH_I \longrightarrow BK.$$

- In fact, something stronger is true. The Tits building:

$$X := \operatorname{hocolim}_{\mathcal{C}} K/H_I,$$

is K -equivariantly contractible. The property above follows from the contractibility of X by taking homotopy orbits with respect to K .

- So for example, $LBSU(2)\langle 3 \rangle$ is equivalent to the pushout of two copies of $BS^1 \times BSU(2)$ over $BS^1 \times BS^1$.

Some consequences of the Homotopical Prop.:

(1) A similar homotopy decomposition holds for $BN(T)$ and BW , in terms of $BN_I(T)$ and BW_I resp.

(2) The map

$$BN(T) \longrightarrow BK,$$

is an equivalence for all but a finite set of (bad) primes.

(3) This set of *bad primes* are those that appear as torsion of W . Since $W \subset GL_r(\mathbb{Z})$, for some r , this set is finite.

(4) The cohomology of the Weyl group W is trivial away from the set of bad primes.

(5) (Broto-K., 2002.) The cohomology ring $H^*(BK, \mathbb{F}_p)$ is a finitely generated \mathbb{F}_p -algebra. Its Krüll dimension is the rank of the largest elementary abelian p -gp in K .

(6) (Broto-K., 2002) Given a finite p -group π , there is an equivalence:

$$\coprod_{\varphi \in \text{Rep}(\pi, K)} BC(\varphi)_{\hat{p}} \simeq \text{Map}(B\pi, BK_{\hat{p}}).$$

However, these spaces need not be p -good! (Dwyer). An algebraic version of the above, using the Lannes T-functor also holds.

- Other homotopy decompositions like the Centralizer decomposition of $BK_{\hat{p}}$, also hold, but may fail to be perfect.

(7) (Broto-K., 2002) Let \mathcal{A} denote the category of elementary abelian subgroups of K . Then the Quillen map:

$$H^*(BK, \mathbb{F}_p) \longrightarrow \lim_{E \in \mathcal{A}} H^*(BE, \mathbb{F}_p),$$

has a nilpotent kernel and cokernel.

What we do (not) know:

(I) W -invariants:

Consider the map $\varphi : K/T \longrightarrow BT$. In cohomology with coefficients in a field \mathbb{F} , Kac shows that the kernel of φ^* is generated by a regular sequence:

$$I = \langle \sigma_1, \dots, \sigma_r \rangle,$$

where $r \leq rk(T)$, and $r = rk(T)$ if $\text{char}(\mathbb{F}) > 0$. The ideal I is called *The ideal of generalized invariants* of K . It contains the Weyl-group invariants ($\text{char}(\mathbb{F}) \neq 2$).

- The ideal I can be defined algebraically in terms of the action of an algebra called *Nil-Hecke algebra* on $H^*(BT, \mathbb{F})$. The Nil-Hecke algebra is a deformation of the group algebra of W .

- Kac shows that If $\text{char}(\mathbb{F}) = p$, then I agrees with the ideal generated by the W -invariants if and only if

$$|W \bmod p| = \prod_i |\sigma_i|, \quad (1)$$

where $|\sigma_i| = \frac{1}{2} \deg(\sigma_i)$, and $|W \bmod p|$ denotes the order of the image of W in $GL_r(\mathbb{F})$.

- The above condition does not require $|W \bmod p|$ to be prime to p . If that does hold, then by a theorem of Chevalley-Shepard-Todd, we can show a stronger statement:

$$H^*(BT, \mathbb{F}_p)^W = \mathbb{F}_p[\sigma_1, \dots, \sigma_r].$$

- **Question:** Does Kac's condition (1) hold away from the bad primes? How is this related to the image of $H^*(BK, \mathbb{F})$ in $H^*(BT, \mathbb{F})^W$?

(II) The homology ring $H_*(K, \mathbb{F}_p)$:

Next, consider the fibration $K \longrightarrow K/T \longrightarrow BT$. Let S denote the image of $H^*(BT, \mathbb{F})$ in $H^*(K/T, \mathbb{F})$.

- Kac-Peterson show that $H^*(K/T, \mathbb{F})$ is a free S -module.

- Their work shows that the Eilenberg-Moore spectral sequence collapses to give an extension of *Hopf Algebras*:

$$1 \longrightarrow H^*(K/T; \mathbb{F}) \otimes_S \mathbb{F} \longrightarrow H^*(K, \mathbb{F}) \longrightarrow \Lambda(y_1, \dots, y_r) \longrightarrow 1,$$

where y_i are odd classes in degree $\deg(\sigma_i) - 1$.

- The even Hopf-algebra $\Gamma_p := H^*(K/T, \mathbb{F}_p) \otimes_S \mathbb{F}_p$ is locally finite over the Steenrod algebra and so it is not finitely generated.

- The dual: Γ_p^* , is a finitely generated algebra. In particular $H_*(K, \mathbb{F}_p)$ is finitely generated.
- Recall that $H^*(K/T, \mathbb{F})$ has a basis: $\{\delta_w\}$, with $w \in W$.

There is a coproduct on $H^*(K/T, \mathbb{F})$ (due to D.Peterson), that induces the coproduct on $\Gamma = H^*(K/T, \mathbb{F}) \otimes_{\mathcal{S}} \mathbb{F}$:

$$\Delta(\delta_w) = \sum_{u*v=w} \delta_u \otimes \delta_v,$$

where $u * v$ is defined to be uv if the lengths add, and is not defined otherwise.

- Since W may be highly non-commutative, the above formula suggests that Γ^* is rarely commutative.

The Example of $K(a)$ (K., 1998):

Let $K = K(a)$. Then the Weyl group W is the free product of two reflections

$$W = \langle r_1, r_2 \mid r_1^2 = r_2^2 = 1. \rangle$$

The extension describing $N(T)$ depends on a however. Notice that the only bad prime is $p = 2$.

- Let k be the smallest nonzero integer so that p divides an element of the sequence f_k . Notice that if $a = 2$, then $k = p$. The generalized invariants over \mathbb{F}_p are generated in degrees 4 and $2k$.
- $H^*(BK, \mathbb{F}_p) = \mathbb{F}_p[x_4, x_{2k}] \otimes E(x_{2k+1})$. There is a non-trivial higher Bockstein with target x_{2k+1} . The polynomial generators detect the generalized invariants in $H^*(BT, \mathbb{F}_p)$.

- The W invariants agree with the generalized invariants for odd primes. For the prime 2, this may not be the case. For example, if a is even, then the action of W on $H^*(BT, \mathbb{F}_2)$ is trivial.

- One may calculate the Hopf algebra $H_*(K(a), \mathbb{F}_p)$. There is an extension of Hopf algebras:

$$1 \rightarrow E(x_3, x_{2k-1}) \rightarrow H_*(K(a), \mathbb{F}_p) \rightarrow \mathbb{F}_p[x_{2k}] \rightarrow 1.$$

This splits as Hopf algebras for odd primes p .

- Rationally, one has isomorphisms:

$$H^*(K(a)/T, \mathbb{Q}) = \mathbb{Q}[x, y]/\langle x^2 + y^2 - a xy \rangle,$$

$$H^*(K(a), \mathbb{Q}) = E(x_3).$$

(III) Other W -modules:

In general, one wants to study the cohomology groups:

$$H^*(W, H^*(K/T, \mathbb{F}_p)).$$

- The W -invariants on $H^*(K/T, \mathbb{R})$ are known for any ring \mathbb{R} , where 2 is not a zero divisor:

$$H^*(K/T, \mathbb{R})^W = \mathbb{R}.$$

- If p is not a bad prime, then:

$$H^i(W, H^*(K/T, \mathbb{F}_p)) = 0, \quad i > 0.$$

The fact that $K/N(T)$ is acyclic away from the bad primes, also follows from the above statement.

(IV) The Stable Transfer:

Consider the map: $BN(T) \longrightarrow BK$. This map is an equivalence away from the bad primes. At bad primes however, the fiber $K/N(T)$ is not \mathbb{F}_p -finite. There is still reason to conjecture:

- **Conjecture:** BK is a stable retract of $BN(T)$.

This conjecture is true for the groups $K(a)$.

Note that each BH_I is a retract of $BN_I(T)$. Now BK and $BN(T)$ are homotopy colimits of the spaces BH_I and $BN_I(T)$ resp., so it is reasonable to attempt to construct a compatible family of transfers. Up to homotopy, this is true. The higher obstructions to doing this sometimes vanish for trivial reasons as in the case of the groups $K(a)$.

(V) Homotopy Uniqueness:

It is known from the work of D. Notbohm that two compact Lie groups G, H are isomorphic if and only if BG and BH are homotopy equivalent. This statement fails for Kac-Moody groups.

J. Aguade and A. Ruiz have shown that there exist distinct Kac-Moody groups $K(a, b)$, such that the spaces $BK(a, b)$ are homotopy equivalent. The family $\{(a, b)\}$ for which this happens is finite. This leads us to the

- **Question:** Given a Kac-Moody group K , is the set of isomorphism classes of Kac-Moody groups K_α for which $BK_\alpha \simeq BK$ a finite set? What about at a prime p ? Or even better, can we define p -compact Kac-Moody groups?

(VI) Representation theory:

Recall that the representation theory of K is a combinatorial problem in gluing the representations of H_I together. These representations tend to be infinite dimensional, and their characters (if defined) are interesting W -invariant formal expressions.

- There is a W -invariant cone in $Lie(T)^*$, called the Tits cone, where this problem has been solved by Kac. More precisely, one can classify all reasonable K representations whose characters belong to this cone. These are called the *Highest weight representations of K* , or the positive energy representations (in the case of the Loop group).
- **Question:** Are there other families of representations?

An example of the Denominator formula:

For the Kac-Moody group $K(2)$, there exist two distinguished characters u, v (given by the two simple roots).

- The formula of Kac on the character of the *trivial* Highest weight representation reduces to the Jacobi Triple-product identity:

$$\frac{\sum_{m \in \mathbb{Z}} (-1)^m u^{m(m-1)/2} v^{m(m+1)/2}}{\prod_{n=1}^{\infty} (1 - u^n v^n)(1 - u^n v^{n-1})(1 - u^{n-1} v^n)} = 1$$

- The left hand side is indeed invariant under the Weyl group $W = \langle r_1, r_2 \mid r_1^2 = r_2^2 = 1 \rangle$, where the action of r_1 and r_2 is given by:

$$r_1(u) = u^{-1}, r_1(v) = v u^2, \quad r_2(u) = u v^2, r_2(v) = v^{-1}.$$

(VII) Dominant K-theory (K., 2008):

If K satisfies a property called *symmetrizable*, then each Highest weight representation is unitary. Consider the universal Hilbert space \mathcal{H} containing each irreducible Highest weight representation infinitely often. Let $F(\mathcal{H})$ denote the space of Fredholm operators on \mathcal{H} in a suitable topology. Then K acts continuously on $F(\mathcal{H})$.

- One defines *Dominant K-theory* for space X with a proper K -action by:

$$\mathbb{K}_K^0(X) = [X, F(\mathcal{H})]_K.$$

- For the universal space \underline{EK} of proper K -actions, we can calculate $\mathbb{K}_K^*(X)$, to get:

- **Theorem:** Let K be such that the category \mathcal{C} consists of *all proper subsets* of \underline{n} , then:

$$\tilde{\mathbb{K}}_K^{n-1}(\underline{EK}) = \mathbb{Z}\langle HW R \rangle.$$

where the right hand side denotes the free abelian group on the set of all irreducible Highest weight representations.

- For central extensions of Loop groups $\tilde{L}G$, the space \underline{EK} is the affine space of principal G -connections over S^1 . In this case, the above theorem follows from a theorem of Freed-Hopkins-Teleman.

- The geometric description of the above isomorphism given by FHT (using a natural equivariant family of Dirac operators) in the case of a Loop group, also holds in the above theorem.

- For any compact subgroup $H < K$, taking $X = K/H$, one gets the Dominant representation ring:

$$DR(H) := \mathbb{K}_K^0(K/H) \subseteq R(H),$$

where $DR(H)$ can be identified with the subring of the representation ring $R(H)$, generated by those H -representations that appear in \mathcal{H} .

- There is a one dimensional rep. $L \in DR(H)$, so that:

$$R(H) = DR(H)[L^{-1}].$$

In particular, if H is a connected, semi-simple compact Lie group, then $DR(H) = R(H)$.

- For arbitrary compact $H < K$, it follows that the completion of $DR(H)$ and $R(H)$ at the augmentation ideal are isomorphic. This gives us:

- **Theorem:** The natural map:

$$\mathbb{K}_K^*(X) \longrightarrow K^*(EK \times_K X),$$

is given by completion with respect to the augmentation ideal on single (proper) orbits.

- Notice however, that the augmentation ideal is not well defined globally since the Highest weight representations are generally infinite dimensional.

- For finite K -CW complexes X , I believe that there is a natural global topology on $\mathbb{K}_K^*(X)$ so that the above map is given by completion. In other words, the Atiyah-Segal completion theorem holds.

The above applies to the spaces \underline{EK} , which are finite K -CW complexes, and so we get a computation of $K^*(BK)$.

(VIII) A rank 3 example:

There is a Kac-Moody group K , with $n = 3$ and with the semisimple parts $L_I \subset H_I$ are:

$$L_{\{1,2\}} = SU(3), \quad L_{\{1,3\}} = SU(2) \times SU(2), \quad L_{\{1\}} = SU(2),$$

where $L_{\{1\}}$ includes as the first factor in $L_{\{1,3\}}$, and by the standard inclusion in $L_{\{1,2\}}$.

- The Weyl group of the above Kac-Moody group is isomorphic to $PGL_2(\mathbb{Z})$, which can therefore be written as an amalgam of Σ_3 and $\mathbb{Z}/2 \times \mathbb{Z}/2$ over $\mathbb{Z}/2$.
- The group $K(2)$ is a (non-compact) subgroup of K , generated by $L_{\{2\}}$ and $L_{\{3\}}$. Hence, the group D_∞ sits naturally inside $PGL_2(\mathbb{Z})$, generated by those two reflections.

Open Questions:

- The structure of the generalized invariants has hardly been studied.
- The image of $H^*(BK, \mathbb{F}_p) \subseteq H^*(BT, \mathbb{F}_p)^W$ is not known.
- The question of homotopy uniqueness of BK and its relation to $BN(T)$ is completely open.
- The stable homotopy type of BK is not understood.
- Can one define “p-compact” versions of Kac-Moody groups? We will need to study (non-finite) p-adic pseudo-reflection groups. This was done for groups generated by two reflections by Aguade-Broto-Saumell.