

ON THE TOPOLOGY OF KAC-MOODY GROUPS

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ABSTRACT. We study the topology of spaces related to Kac-Moody groups. Given a split Kac-Moody group over \mathbb{C} , let K denote the unitary form with maximal torus T having normalizer $N(T)$. In this article we study the (co)homology of K as a Hopf algebra. In particular, if \mathbb{F} has positive characteristic, we show that $H_*(K, \mathbb{F})$ is a finitely generated algebra, and that $H^*(K, \mathbb{F})$ is finitely generated only if K is a compact Lie group. We also study the stable homotopy type of the classifying space BK and show that it is a retract of the classifying space $BN(T)$ of $N(T)$. We illustrate our results with the example of rank two Kac-Moody groups.

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1. INTRODUCTION

In this paper, we deal with a class of topological groups known as Kac-Moody groups [K2]. By a Kac-Moody group, we shall mean the unitary form of a split Kac-Moody group over \mathbb{C} . We refer the reader to [Ku] for a beautiful treatment of the subject. These groups

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form a natural extension of the class of compact Lie groups, and share many of their properties. They are known to contain the class of (polynomial) loop groups, which go by the name of affine Kac-Moody groups. With the exception of compact Lie groups, Kac-Moody groups over \mathbb{C} are not even locally compact. Hence geometric arguments used to study the topology of compact Lie groups and their flag varieties no longer extend to general Kac-Moody groups. This led Kac-Peterson to construct of a whole new set of techniques applicable in this context. Underlying these techniques is a collection of (annihilation) operators that generate a ring \mathcal{O} [K1, KK] given by a deformation of the group ring of the Weyl group. The cohomology rings of the flag varieties admit an action of \mathcal{O} which intertwines the action of the Weyl group on the flag varieties. This induces a very rich structure that can be exploited to prove various structure theorems about flag varieties [Ki, K1, Ku], as well as the Kac-Moody group itself [K1]. Further techniques were developed by the author in [Ki] in order to study the classifying spaces of these groups.

The purpose of this paper is two fold: Firstly, we will give independent proofs of various homological structure theorems, many of which are known to the experts, but whose proofs are either absent in print or exist in weaker generality. Some of these results were stated in [K1], with proofs to appear in a later publication that never made it to print. Also included in this paper are new results regarding the structure of the (co)homology rings of Kac-Moody groups and their flag varieties with coefficients in a field of positive characteristic. Secondly, in this paper we also study the classifying space of a Kac-Moody group. In particular, we will show that the stable homotopy type of the classifying space is a retract of the classifying space of the normalizer of the maximal torus.

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2. BACKGROUND AND STATEMENT OF NEW RESULTS

Kac-Moody groups have been extensively studied and much is known about their general structure, representation theory and topology [K1, K2, K3, Ki, Ku, KW, T] (see [Ku] for a modern prespective). Their construction begins with a finite integral matrix $A = (a_{i,j})_{i,j \in I}$ with the properties that $a_{i,i} = 2$ and $a_{i,j} \leq 0$ for $i \neq j$. Moreover, we demand that $a_{i,j} = 0$ if and only if $a_{j,i} = 0$. These conditions define a *Generalized Cartan Matrix*.

Given a generalized Cartan matrix A , one may construct a complex Lie algebra $\mathfrak{g}(A)$ using the Harishchandra-Serre relations. This Lie algebra contains a finite dimensional Cartan subalgebra \mathfrak{h} that admits an integral form $\mathfrak{h}_{\mathbb{Z}}$ and a real form $\mathfrak{h}_{\mathbb{R}} = \mathfrak{h}_{\mathbb{Z}} \otimes \mathbb{R}$. The lattice $\mathfrak{h}_{\mathbb{Z}}$ contains a finite set of primitive elements $h_i, i \in I$ called "simple coroots". Similarly, the dual lattice $\mathfrak{h}_{\mathbb{Z}}^*$ contains a special set of elements called "simple roots" $\alpha_i, i \in I$. One may decompose $\mathfrak{g}(A)$ under the adjoint action of \mathfrak{h} to obtain a triangular form as in the classical theory of semisimple Lie algebras. Let η_{\pm} denote the positive and negative "nilpotent"

subalgebras respectively, and let $\mathfrak{b}_\pm = \mathfrak{h} \oplus \eta_\pm$ denote the corresponding ‘‘Borel’’ subalgebras. The structure theory of $\mathfrak{g}(A)$ leads to a construction (in much the same way that Chevalley groups are constructed), of a topological group $G(A)$ called the (minimal, split) Kac-Moody group over the complex numbers. The group $G(A)$ supports a canonical anti-linear involution ω , and one defines the unitary form $K(A)$ as the fixed group $G(A)^\omega$. It is the group $K(A)$ that we refer to as the Kac-Moody group in this article.

Given a subset $J \subseteq I$, one may define a parabolic subalgebra $\mathfrak{g}_J(A) \subseteq \mathfrak{g}(A)$ generated by \mathfrak{b}_+ and the root spaces corresponding to the set J . For example, $\mathfrak{g}_\emptyset(A) = \mathfrak{b}_+$. One may exponentiate these subalgebras to parabolic subgroups $G_J(A) \subset G(A)$. We then define the unitary Levi factors $K_J(A)$ to be the groups $K(A) \cap G_J(A)$. Hence $K_\emptyset(A) = T$ is a torus of rank $2|I| - rk(A)$, called the maximal torus of $K(A)$. The normalizer $N(T)$ of T in $K(A)$, is an extension of a discrete group $W(A)$ by T . The Weyl group $W(A)$ has the structure of a crystallographic Coxeter group generated by reflections $r_i, i \in I$. $W(A)$ has a Coxeter presentation given as follows:

$$W(A) = \langle r_i, i \in I \mid r_i^2 = 1, (r_i r_j)^{m_{ij}} = 1 \rangle,$$

where m_{ij} are suitable integers [H]. For $J \subseteq I$, let $W_J(A)$ denote the subgroup generated by the corresponding reflections $r_j, j \in J$. The group $W_J(A)$ is a crystallographic Coxeter group in its own right that can be identified with the Weyl group of $K_J(A)$.

Given a generalized Cartan matrix $A = (a_{i,j})_{i,j \in I}$, define a category $\mathcal{S}(A)$ to be the poset category (under inclusion) of subsets $J \subseteq I$ such that $K_J(A)$ is a compact Lie group. This is equivalent to demanding that $W_J(A)$ is a finite group. Notice that $\mathcal{S}(A)$ contains all subsets of I of cardinality less than two. In particular, $\mathcal{S}(A)$ is nonempty and has an initial object given by the empty set. The category $\mathcal{S}(A)$ is also known as the poset of spherical subsets. The topology on the group $K(A)$ is the strong topology generated by the compact subgroups $K_J(A)$ for $J \in \mathcal{S}(A)$ (See Appendix). More precisely, $K(A)$ is the amalgamated product of the compact Lie groups $K_J(A)$, in the category of topological groups. For an arbitrary subset $L \subseteq I$, the topology induced on homogeneous space of the form $K(A)/K_L(A)$ makes it into a CW-complex, with only even cells, indexed by the set of cosets $W(A)/W_L(A)$.

Let $K(A)/T \xrightarrow{\psi} BT$ be the map that classifies the principal T -bundle $K(A) \xrightarrow{\pi} K(A)/T$. Let \mathbb{F} be a field and let $J \subset H^*(BT, \mathbb{F})$ denote the ideal given by the kernel of ψ^* . It has been shown in [K] that J is generated by a regular sequence $\langle \sigma_1, \dots, \sigma_r \rangle$, with $r \leq rank(T)$. The ideal J is called the ideal of *Generalized Invariants*. Let $S \subseteq H^*(K(A)/T, \mathbb{F})$ denote the subring generated by the image of ψ^* . It is also shown in [K] that $H^*(K(A)/T, \mathbb{F})$ is a free S -module.

From the CW structure of $K(A)/T$, we notice that $H^*(K(A)/T, \mathbb{Z})$ has a (Schubert) basis: $\{\delta^w\}$, with $w \in W(A)$. This extends to a basis in cohomology with arbitrary coefficients. Let us define a coproduct (introduced by D. Peterson) on the cohomology of $K(A)/T$:

$$\Delta(\delta^w) = \sum_{uv=w} \delta^u \otimes \delta^v,$$

where the sum is taken over all reduced expressions for w i.e. expressions where the minimal word length of w with respect to the generators r_i equals the sum of the minimal word lengths of u and v .

We will prove the following theorem:

Theorem 2.1. *The image in cohomology of the projection map $K(A) \xrightarrow{\pi} K(A)/T$ is isomorphic to $H^*(K(A)/T, \mathbb{F}) \otimes_S \mathbb{F}$. Moreover, this image is a sub Hopf algebra with the coalgebra structure induced via the coproduct Δ defined above.*

Remark 2.2. *The statement of the above theorem was communicated to the author by D. Peterson. The first part of this theorem has been proved in [K]. The second part is stated there without proof. In [K], V. Kac shows that $H^*(K(A), \mathbb{F})$ is free over the image of π^* , and there is a short exact sequence of algebras:*

$$1 \rightarrow H^*(K(A)/T, \mathbb{F}) \otimes_S \mathbb{F} \xrightarrow{\pi^*} H^*(K(A), \mathbb{F}) \longrightarrow \Lambda(x_1, \dots, x_r) \rightarrow 1.$$

Here $\Lambda(x_1, \dots, x_r)$ denotes an exterior algebra on classes x_i of degree given by $\deg(\sigma_i) - 1$. The number of exterior generators is bounded by the rank of T , and equals it if \mathbb{F} has positive characteristic. In light of the previous theorem, this extension is actually an extension of Hopf algebras. In addition, the elements x_i can be chosen to be primitive [MM].

Next, will study the homology ring $H_*(K(A), \mathbb{F})$ and prove:

Theorem 2.3. *Assume that \mathbb{F} has positive characteristic. Then the Pontrjagin ring $H_*(K(A), \mathbb{F})$ is a finitely generated algebra. In addition, $H^*(K(A), \mathbb{F})$ and $H^*(K(A)/T, \mathbb{F})$ are finitely generated if and only if $K(A)$ is a compact Lie group.*

Remark 2.4. *The Pontrjagin ring $H_*(K(A), \mathbb{F})$ will in general be highly non commutative. The structure of the rational Pontrjagin ring: $H_*(K(A), \mathbb{Q})$ for a general Kac-Moody group remains unclear to the author.*

Now let us consider the classifying space of $K(A)$, denoted by $BK(A)$. The study of this space was begun in [Ki] and continued in [BK]. It was shown that $BK(A)$ can be described decomposed in terms of the classifying space $BK_J(A)$, as J varied through the poset $\mathcal{S}(A)$. A similar result was proved for the spaces $BN(T)$ and $BW(A)$. In this paper, we will use this homotopy decomposition to prove:

Theorem 2.5. *Let $BN(\mathbf{T})_+$ and $BK(\mathbf{A})_+$ denote the suspension spectra of the spaces $BN(T)_+$ and $BK(A)_+$ respectively (each endowed with a disjoint base point). Then the canonical map $BN(T)_+ \rightarrow BK(A)_+$ admits a stable retraction $BK(\mathbf{A})_+ \rightarrow BN(\mathbf{T})_+$.*

And finally, in the appendix we will study the topology on $K(A)$ and $BK(A)$. In particular, we will show that $K(A)$ is the colimit of the compact Lie groups $K_J(A)$ in the category of topological groups. We will also prove that $BK(A)$ has the homotopy type of a CW complex.

3. COHOMOLOGY AND INTEGRATION ALONG THE FIBER

In the following sections we suppress reference to the generalized Cartan matrix and let K denote a Kac-Moody group with compact Levi factors K_J , and maximal torus T . Let W denote the Weyl group. In this section, we will work with cohomology with coefficients in some arbitrary ring R .

Recall the coroots $h_i \in \mathfrak{h}_{\mathbb{Z}}$. Let $h_i^* \in \mathfrak{h}_{\mathbb{Z}}^*$ denote the dual characters with the property $h_i^*(h_j) = \delta_{ij}$, where δ_{ij} denotes the Kronecker symbol. The elements h_i^* need not be unique,

but we fix a choice throughout. We shall use the same notation to denote the Euler class $h_i^* \in H^2(BT)$ of the line bundle $ET \times_T \mathbb{C}$ over BT , where the action of T on \mathbb{C} is obtained by exponentiating h_i^* . It is not hard to show that the set $\{h_i^*\}$, can be extended to an R-basis of $H^2(BT)$.

Recall the map $K/T \xrightarrow{\psi} BT$ classifying the principal bundle $K \xrightarrow{\pi} K/T$. The homomorphism $\psi^* : H^*(BT) \rightarrow H^*(K/T)$ is called the *Characteristic Homomorphism*.

Claim 3.1. $\psi^* : H^2(BT) \rightarrow H^2(K/T)$ has the property: $\psi^*(h_i^*) = \delta^{r_i}$, where δ^{r_i} denotes elements of the Schubert basis of $H^*(K/T)$ corresponding to the generating set $r_i \in W$.

Proof. The proof follows from the CW decomposition of K/T . By the construction of Kac-Moody groups, there exist injective homomorphisms: $\varphi_i : \text{SU}_2 \rightarrow K_i \subset K$ extending the subgroups $S^1 \subset T$ obtained by exponentiating the coroot h_i . The induced map on the level of flag varieties: $\text{SU}_2/S^1 \rightarrow K/T$ is simply the inclusion of the cell corresponding to the Weyl element r_i . It follows that the restriction of L_j to SU_2/S^1 via $\psi \circ \varphi_i : \text{SU}_2/S^1 \rightarrow BT$ is null if $i \neq j$, and is the Hopf bundle if $i = j$. It is easy to see that this implies that: $\psi^*(h_i^*) = \delta^{r_i}$ completing the proof. \square

We now proceed to construct certain annihilation operators $A_i, i \in I$ acting on $H^*(K/T)$ and $H^*(BT)$ that were mentioned in the introduction. Let us first introduce the relevant framework:

Definition 3.2. Let H^* denote cohomology with coefficients in a ring R which will be fixed throughout this section. An oriented fibration is a triple (π, n, τ) where:

- (1) $F \rightarrow E \xrightarrow{\pi} B$ is a Serre fibration
- (2) $H^i(F) = 0$ for $i > n$ and $\pi_1(B)$ acts trivially on $H^n(F)$.
- (3) $\tau : H^n(F) \rightarrow R$ is a homomorphism of R-modules.

A morphism $(f, g, h) : (\pi, n, \tau) \rightarrow (\pi', n', \tau')$ will be a commutative diagram:

$$\begin{array}{ccc} F & \xrightarrow{f} & F' \\ \downarrow & & \downarrow \\ E & \xrightarrow{g} & E' \\ \downarrow & & \downarrow \\ B & \xrightarrow{h} & B' \end{array}$$

such that $n = n'$ and $\tau \circ f^* = \tau'$.

Given an oriented fibration (π, n, τ) . we can define a homomorphism of R-modules:

$$\int_{\tau} : H^*(E) \longrightarrow H^{*-n}(B)$$

called *Integration along the fiber* as follows: Consider the Serre spectral sequence for the fibration $F \rightarrow E \xrightarrow{\pi} B$. Then \int_{τ} is defined as the composite:

$$H^*(E) \twoheadrightarrow E_{\infty}^{*-n, n} \longrightarrow E_2^{*-n, n} = H^{*-n}(B; H^n(F)) \xrightarrow{\tau^*} H^{*-n}(B).$$

Theorem 3.3. (Properties of integration along the fiber)

- (A) If (π, n, τ) is an oriented fibration, then $\int_{\tau} : H^*(E) \rightarrow H^{*-n}(B)$ is a map of $H^*(B)$ -modules where $H^*(E)$ is an $H^*(B)$ -module via π^* .
- (B) Given a morphism $(f, g, h) : (\pi, n, \tau) \rightarrow (\pi', n', \tau')$ we have $h^* \circ \int_{\tau'} = \int_{\tau} \circ g^*$.
- (C) For an oriented spherical fibration $S^n \rightarrow E \rightarrow B$ with τ given by the fundamental class in $H_n(S^n) = \text{Hom}(H^n(S^n); R)$, the homomorphism \int_{τ} is given by the composite:

$$H^*(E) \xrightarrow{\delta} H^{*+1}(D; E) \xrightarrow{T} H^{*-n}(B)$$

where $D = \text{cofiber}(\pi)$ is the disk bundle corresponding to the sphere bundle E . Here δ denotes the boundary homomorphism for the pair (D, E) and T is the Thom isomorphism.

- (D) For an oriented fibration (π, n, τ) we have $\int_{\tau} \circ \pi^* \circ \int_{\tau} = 0$ if $n > 0$.

Proof. Part (A) is obvious using the multiplicative structure of the Serre spectral sequence. Part (B) follows easily from the naturality of the spectral sequence. For part (C), first notice that the statement is obvious if B is a point. Now use the fact that both homomorphisms are module homomorphisms over $H^*(B)$. Finally for part (D) notice first that the composite $\pi^* \circ \int_{\tau}$ is detected on $E_{\infty}^{*,0}$ and use the edge homomorphism. □

Definition 3.4. Given $i \in I$, consider the fiber bundle: $K_i/T \rightarrow K/T \xrightarrow{\pi_i} K/K_i$. Note that K_i/T may be canonically identified with $\mathbb{C}P^1$ via the homeomorphism $\varphi_i : \text{SU}_2/S^1 \rightarrow K_i/T$. Let $\tau_i \in \text{Hom}(H^2(K_i/T); R) = \text{Hom}(H^2(\mathbb{C}P^1); R) = H_2(\mathbb{C}P^1)$ be the fundamental class. Note that $(\pi_i, 2, \tau_i)$ is an oriented fibration. Define operators:

$$A_i : H^*(K/T) \longrightarrow H^{*-2}(K/T) \quad A_i = \pi_i^* \circ \int_{\tau_i}$$

Similarly for the bundle $K_i/T \rightarrow BT \xrightarrow{\theta_i} BK_i$, we define operators by the same name:

$$A_i : H^*(BT) \longrightarrow H^{*-2}(BT) \quad A_i = \theta_i^* \circ \int_{\tau_i}$$

where θ_i is the inclusion of the maximal torus.

Remark 3.5. Note that the operators A_i are natural with respect to ring homomorphisms $R \rightarrow R'$.

Now we have a commutative diagram:

$$\begin{array}{ccc} K_i/T & \xrightarrow{=} & K_i/T \\ \downarrow & & \downarrow \\ K/T & \xrightarrow{\psi} & BT \\ \downarrow & & \downarrow \\ K/K_i & \longrightarrow & BK_i \end{array}$$

From property (B) 3.3 we derive:

$$\psi^* \circ A_i = A_i \circ \psi^*.$$

4. THE SCHUBERT BASIS AND THE NIL HECKE RING

Recall there is a CW-decomposition [Ku] of K/K_J , for $J \subseteq I$ as:

$$K/K_J = \coprod_{w \in W/W_J} BwB/B = \coprod_{w \in W/W_J} BwG_J/G_J,$$

where $B = G_\emptyset$ is the Borel subgroup, and G_J are the parabolic subgroups. The space BwB/B is a subspace homeomorphic to $\mathbb{C}^{l(w)}$, where $l(w)$ denotes the word length of the element w in terms of the generators r_i . Hence $H^*(K/K_J)$ has an \mathbb{R} -basis $\{\delta^w\}_{w \in W/W_J}$ where $\delta^w \in H^{2l(w)}(K/K_J)$ represents the affine cell BwB/B . The set $\{\delta^w\}$ is called the Schubert basis. We define the Schubert subvarieties of K/K_J as closures of cells:

$$X_w^J = \overline{BwG_J/G_J} = \coprod_{w' \in W/W_J, w' \leq w} Bw'B/B,$$

where we define a partial order on W/W_J by identifying it with the set of minimal coset representatives W^J , and then using the Bruhat order [H]. Notice that $H_{2l(w)}(X_w^J)$ is a free \mathbb{R} -module generated by a class $[X_w^J]$ where $[X_w^J]$ represents the cell: BwB/B . Let $i_w : X_w^J \rightarrow K/K_J$ be the inclusion.

Define homomorphisms $L_w^J : H^*(K/K_J) \rightarrow \mathbb{R}$ via:

$$H^*(K/K_J) \xrightarrow{i_w^*} H^*(X_w^J) \left\langle \begin{array}{c} ; [X_w^J] \\ \longrightarrow \end{array} \right\rangle \mathbb{R}$$

Note that

$$L_w^J(\delta^v) = \begin{cases} 1 & \text{if } v = w \in W/W_J \\ 0 & \text{otherwise} \end{cases}$$

Observe that for $v \in W/W_J$ we have a commutative diagram:

$$\begin{array}{ccc} X_v & \xrightarrow{i_v} & K/T \\ \downarrow p & & \downarrow \pi_J \\ X_v^J & \xrightarrow{i_v} & K/K_J \end{array}$$

and $p_*[X_v] = [X_v^J]$. For $x \in H^*(K/K_J)$ we have the identity:

$$\langle i_v^* \pi_J^*(x); [X_v] \rangle = \langle p^* i_v^*(x); [X_v] \rangle = \langle i_v^*(x); [X_v^J] \rangle.$$

It follows that $L_v^J = L_v \circ \pi_J^*$, where $L_v = L_v^\emptyset$.

Lemma 4.1. *If $v \in W$ such that $l(vr_j) > l(v)$ then $L_{vr_j} = L_v \circ A_j$.*

Proof. Since $l(vr_j) > l(v)$ we have $v \in W^{\{j\}}$ [H]. Note that we have a pullback diagram:

$$\begin{array}{ccc} X_{vr_j} & \xrightarrow{i_{vr_j}} & K/T \\ \downarrow p & & \downarrow \pi_j \\ X_v^j & \xrightarrow{i_v} & K/K_j \end{array}$$

This is because if w is in $uW_{\{j\}}$ for some $u \leq v$, $u \in W^{\{j\}}$, then $w \leq vr_j$.

Now consider the Serre spectral sequence for the fibration: $K_j/T \rightarrow X_{vr_j} \xrightarrow{p} X_v^j$ and notice that it collapses at E_2 . Moreover, the class $[X_{vr_j}]$ is represented by $[\tau_j] \otimes [X_v^j]$ at the E_2 stage. Consequently, we have:

$$\langle x; [X_{vr_j}] \rangle = \left\langle \int_{\tau_j} x; [X_v^j] \right\rangle, \quad x \in H^*(X_{vr_j}).$$

It follows that $L_{vr_j} = L_v^j \circ \int_{\tau_j}$. We now get a sequence of equalities:

$$L_{vr_j} = L_v^j \circ \int_{\tau_j} = L_v \circ \pi_j^* \circ \int_{\tau_j} = L_v \circ A_j$$

where we have used the fact that $L_v^j = L_v \circ \pi_j^*$. This completes the proof. □

Lemma 4.2. *The action of the operators A_j on the Schubert basis is given by:*

$$A_j(\delta^w) = \begin{cases} \delta^{wr_j} & \text{if } l(wr_j) < l(w) \\ 0 & \text{otherwise} \end{cases}$$

Proof. If $w = vr_j$ and $l(v) < l(w)$, then by 4.1 and using

$$L_w^J(\delta^v) = \begin{cases} 1 & \text{if } v = w \in W^J \\ 0 & \text{otherwise} \end{cases}$$

as seen before, we get $A_j(\delta^{vr_j}) = \delta^v$. Now if $l(wr_j) > l(w)$, then from above we have $A_j(\delta^{wr_j}) = \delta^w$ and hence $A_j(\delta^w) = A_j A_j(\delta^{wr_j}) = 0$ by 3.3 part (D). □

The following theorem is crucial in the study of Kac-Moody groups and their flag varieties. This structure was introduced by Kac-Peterson [K1], and studied further by Kostant-Kumar [KK]. The theorem below can be shown to follow from the previous lemma, and the Bruhat exchange relations. We refer the reader to [Ku] for a detailed proof.

Theorem 4.3. [KK] *Define the Nil Hecke ring \mathcal{O} to be the ring generated by the operators A_i , $i \in I$ acting on $H^*(K/T)$. Then we have:*

- (i) \mathcal{O} is generated by A_i , $i \in I$, and relations $A_i^2 = 0$, $A_i A_j A_i \cdots = A_j A_i A_j \cdots$ (m_{ij} factors).
- (ii) \mathcal{O} has an R-basis given by $\{A_w\}_{w \in W}$ where $A_w = A_{i_1} A_{i_2} \cdots A_{i_k}$ is well defined whenever $w = r_{i_1} r_{i_2} \cdots r_{i_k}$ is a reduced (or minimal length) expression. Furthermore,

$$A_w(\delta^v) = \begin{cases} \delta^{vw^{-1}} & \text{if } l(w) + l(vw^{-1}) = l(v) \\ 0 & \text{otherwise} \end{cases}$$

Remark 4.4. Notice that the algebra \mathcal{O} is a deformation of the group algebra of W . It can be shown that it acts on $H^*(BT)$ through the operators A_i . There are versions of this algebra acting on T -equivariant cohomology, as well as versions acting on the (equivariant) K-theory of the space K/T . [Ku].

5. PROPERTIES OF A_i AND THE ACTION OF W

Note that $W = N(T)/T$ acts on T and consequently on BT . Define an action of W on K/T by making $uT \in W$ act on $kT \in K/T$ by $uT \cdot kT = ku^{-1}T$. Consequently, we get actions of W on $H^*(K/T)$ and $H^*(BT)$. It is easy to verify that the characteristic homomorphism $\psi^* : H^*(BT) \rightarrow H^*(K/T)$ commutes with the W -action. In this section, we recall some fundamental results about the structure of the cohomology of Kac-Moody flag varieties as described in [K1, KK, Ku].

Recall the roots $\alpha_i \in \mathfrak{h}_{\mathbb{Z}}^*$. We use the same notation to denote the Euler class $\alpha_i \in H^2(K/T)$ of the line bundles $K \times_T \mathbb{C}$, where T acts on \mathbb{C} by exponentiating the root α_i . Similarly, let $\alpha_i \in H^2(BT)$ denote the Euler class of the line bundle $ET \times_T \mathbb{C}$ under the same action.

Claim 5.1. $\alpha_i \in H^2(K/T)$ is given by $\alpha_i = \sum_{j \in I} a_{ji} \delta^{r_j}$.

Proof. Under the canonical pairing between $\mathfrak{h}_{\mathbb{Z}}^*$ and $\mathfrak{h}_{\mathbb{Z}}$, we have the equality: $\alpha_i(h_j) = a_{ji}$. The proof now follows using Claim 3.1. \square

Claim 5.2. The action of W on $H^2(BT)$ and $H^2(K/T)$ satisfies:

- (a) $r_i(h_j^*) = h_j^* - \delta_{ij} \alpha_i$ where δ_{ij} denotes the Kronecker symbol and $\alpha_i \in H^2(BT)$.
- (b) $r_i(\alpha_j) = \alpha_j - a_{ij} \alpha_i$, with $\alpha_i, \alpha_j \in H^2(BT)$.
- (c) $r_i(\delta^{r_j}) = \delta^{r_j} - \delta_{ij} \alpha_i$, with $\alpha_i \in H^2(K/T)$.

Proof. We may canonically identify $H^2(BT, \mathbb{R})$ with $\mathfrak{h}_{\mathbb{Z}}^* \otimes \mathbb{R}$. The action of W is therefore given by $r_i(\lambda) = \lambda - \lambda(h_i) \alpha_i$. From this formula, part (a) and (b) follow easily. For part (c) we simply invoke Claim 3.1. \square

Recall that the operator A_i was defined via integration along the fiber for the fibration:

$$K_i/T \longrightarrow K/T \xrightarrow{\pi_i} K/K_i$$

Claim 5.3. For any element $x \in H^*(K/T)$, there is a unique pair of elements $y, z \in \text{Im } \pi_i^*$ such that $x = \delta^{r_i} \cup y + z$. Moreover, $y = A_i(x)$ and $z = x - \delta^{r_i} \cup A_i(x)$. In particular, $x \in \text{Im } \pi_i^*$ if and only if $A_i(x) = 0$.

Proof. The Serre spectral sequence for $K_i/T \rightarrow K/T \rightarrow K/K_i$ collapses at E_2 . Since δ^{r_i} restricts to a generator of $H^*(K_i/T)$, the result follows from the multiplicative structure of the spectral sequence. \square

Claim 5.4. One has the relation $\alpha_i \cup A_i(x) = x - r_i(x)$ for $x \in H^*(K/T)$.

Proof. Write x as $x = \delta^{r_i} \cup A_i(x) + z$ with $z \in \text{Im } \pi_i^*$. Since r_i acts on K/T through right multiplication by an element in K_i , it fixes the image of π_i^* . Thus $r_i(x) = (\delta^{r_i} - \alpha_i) \cup A_i(x) + z$ using 5.2. The difference of the above two equations gives the required result. \square

Claim 5.5. If $2 \in R$ is not a zero divisor, then α_i is not a zero divisor on the image of π_i^* . In particular, under this assumption, $A_i(x)$ is the unique element in the image of π_i^* satisfying the equality given in 5.4.

Proof. Assume $\alpha_i \cup y = 0$ for some $y \in \text{Im } \pi_i^*$. Note that $\alpha_i = 2\delta^{r_i} + z$ for some $z \in \text{Im } \pi_i^*$ using 5.1. Thus we have $\delta^{r_i} \cup 2y + z \cup y = 0$. From 5.3 we get $2y = 0$ which implies $y = 0$ by assumption. \square

Remark 5.6. The same proof as above shows that 5.3, 5.4 and 5.5 hold for $H^*(BT)$ once we replace δ^{r_i} by h_i^* , and π_i^* by θ_i^* .

Theorem 5.7. $A_i(u \cup v) = A_i(u) \cup r_i(v) + u \cup A_i(v)$, where u, v are arbitrary homogeneous elements in $H^*(K/T)$ or $H^*(BT)$.

Proof. Since the operators A_i are natural with respect to ring homomorphisms, it is sufficient to prove the theorem for $R = \mathbb{Z}$. Note that:

$$\begin{aligned} \alpha_i \cup (A_i(u) \cup r_i(v) + u \cup A_i(v)) &= (u - r_i(u)) \cup r_i(v) + u \cup (v - r_i(v)) \\ &= u \cup v - r_i(u \cup v) \\ &= \alpha_i \cup A_i(u \cup v) \end{aligned}$$

Thus, by 5.5, we will be done if we can show that $x = A_i(u) \cup r_i(v) + u \cup A_i(v) \in \text{Im } \pi_i^*$. It suffices to show that $\alpha_i \cup A_i(x) = 0$, but we have $\alpha_i \cup A_i(x) = x - r_i(x) = -\alpha_i \cup A_i(u) \cup A_i(v) + \alpha_i \cup A_i(u) \cup A_i(v) = 0$ and so we are done. \square

Theorem 5.8. Let $\mathcal{O}_J \subseteq \mathcal{O}$ be the subalgebra generated by the operators $A_j, j \in J \subseteq I$. Then $H^*(K/K_J) = H^*(K/T)^{\mathcal{O}_J}$, where $H^*(K/T)^{\mathcal{O}_J}$ denotes all elements annihilated by \mathcal{O}_J and $H^*(K/K_J)$ is identified with its image in $H^*(K/T)$ via π_J^* .

Proof. The image of π_J^* is a free R -module on the Schubert basis $\delta^w, w \in W^J$. But $w \in W^J$ if and only if $l(wr_j) > l(w)$, for all $j \in J$. The result follows from the formula for the action of A_j on the Schubert basis. \square

Corollary 5.9. If $2 \in R$ is not a zero divisor, then $H^*(K/K_J) = H^*(K/T)^{W_J}$ is the submodule of W_J -invariant elements for $J \subseteq I$. In particular $H^*(K/T)^W = H^0(K/T) = R$.

Proof. $H^*(K/K_J) = H^*(K/T)^{\mathcal{O}_J} = H^*(K/T)^{W_J}$ using 5.4 and 5.5. \square

Remark 5.10. Let $R = \mathbb{F}_2$ and consider the example of the Kac-Moody group corresponding to the Cartan matrix $\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$. Then $H^*(K/T; \mathbb{F}_2)^W = H^*(K/T; \mathbb{F}_2)$, which show that the assumption that 2 is not a zero-divisor is necessary.

6. THE COHOMOLOGY OF K

In this section we recall some results from [K]. Recall the characteristic homomorphism $\psi^* : H^*(BT) \rightarrow H^*(K/T)$. We begin by identifying the kernel of ψ^* . Let $H^*(BT)^+$ denote homogeneous elements of $H^*(BT)$ of positive degree. Let $J \subset H^*(BT)$ be defined as: $J = \{u \in H^*(BT) \mid A_{i_1} A_{i_2} \cdots A_{i_k}(u) \in H^*(BT)^+ \forall i_1, \dots, i_k\}$. We have:

Theorem 6.1. [K] $J = \text{Ker } \psi^*$.

Proof. Notice first that $z \in H^*(K/T)^+$ is nonzero if and only if there exists a sequence $i_1 \dots i_k$ such that $0 \neq A_{i_1} \cdots A_{i_k}(z) \in H^*(K/T)$. Now since ψ^* is an isomorphism in degree 0, the statement $A_{i_1} \cdots A_{i_k}(u) \in H^*(BT)^+ \forall i_1 \dots i_k$ is equivalent to the statement $\psi^* A_{i_1} \cdots A_{i_k}(u) \in H^*(K/T)^+ \forall i_1 \dots i_k$, i.e. $A_{i_1} \cdots A_{i_k} \psi^*(u) \in H^*(K/T)^+ \forall i_1 \dots i_k$ which is equivalent to $\psi^*(u) = 0$. \square

Remark 6.2. The ideal J is known as the ideal of Generalized Invariants of W . It has been studied in detail and appears to be of independent interest [K4].

For the rest of this section we assume that the ring \mathbb{R} is a field \mathbb{F} .

Lemma 6.3. [K] J/J^2 is a free $H^*(BT)$ -module.

Proof. Let y_1, y_2, \dots be a set of homogeneous elements of J such that y_1, y_2, \dots form a minimal set of generators of J/J^2 as an $H^*(BT)$ -module ordered by increasing degree: $0 < \deg(y_1) \leq \deg(y_2) \leq \deg(y_3) \dots$. Note that the operators A_i preserve J^2 and that:

$$A_i(y_k) = \sum_{j < k} r_j y_j + J^2, \quad (*)$$

with $r_j \in H^*(BT)$. Let $\sum_{j < k} s_j y_j \in J^2$ be some homogeneous relation. We can assume $s_k \notin J$. Choose a sequence $i_1 \dots i_k$ such that $0 \neq A_{i_1} \dots A_{i_k}(s_k) \in \mathbb{F}$. Applying $A_{i_1} \dots A_{i_k}$ to the relation and using $(*)$ repeatedly we notice that:

$$y_k = \sum_{j < k} t_j y_j + J^2 \quad t_j \in H^*(BT)$$

which is a contradiction to the minimality of the set of generators. \square

Theorem 6.4. [K] J is generated by a regular sequence $J = \langle \sigma_1, \sigma_2, \dots, \sigma_r \rangle$, where $r \leq \text{rank}(T)$.

Proof. First notice that any regular sequence in $H^*(BT)$ must have length $\leq \text{rank}(T)$. So it remains to show that J is generated by a regular sequence. But this follows from a theorem of Vasconcelos [V] which says that for a graded algebra A of finite global dimension a homogeneous ideal $J \subseteq A$ is generated by a regular sequence if and only if J/J^2 is a free A/J -module. \square

Remark 6.5. If \mathbb{F} has positive characteristic p , the length of this sequence is exactly $\text{rank}(T)$. This can be seen as follows: First notice that if $\lambda \in H^*(BT)^W$, then its p -th power λ^p is annihilated by all the operators A_i . Now notice that $H^*(BT)^W$ contains the Dickson invariants defined as the invariants with respect to the action of $GL_n(\mathbb{F})$, where n is the rank of T . The Dickson invariants (or their p -th powers) form a regular sequence of length n . Hence J contains a regular sequence of maximal length and must therefore itself be generated by a sequence of maximal length.

Define a subring S of $H^*(K/T)$ via $S = \text{Im}\{\psi^* : H^*(BT) \rightarrow H^*(K/T)\} \cong H^*(BT)/J$. Notice that:

- S is the subring of $H^*(K/T)$ generated by $H^2(K/T)$.
- If \mathbb{F} has characteristic $p > 0$ then S is a finite dimensional vector space over \mathbb{F} . In fact, S is a Poincaré duality algebra.
- Let $2d_i$ be the degree of σ_i , and let n be the rank of T , then the Poincaré series of S is:

$$P_t(S) = \frac{\prod_{i=1}^r (1 - t^{2d_i})}{(1 - t^2)^n}.$$

Theorem 6.6. [K] $H^*(K/T)$ is a free S -module.

Proof. Proceed as before. Let $y_1, y_2 \dots$ be a set of homogeneous elements of $H^*(K/T)$ so that $y_1, y_2 \dots$ form an \mathbb{F} -basis of $H^*(K/T) \otimes_S \mathbb{F}$, and $0 = \deg(y_1) \leq \deg(y_2) \leq \deg(y_3) \dots$. It is clear that the y_i generate $H^*(K/T)$ as an S -module. Note that the operators A_i preserve S for all i and

$$A_i(y_k) = \sum_{j < k} r_j y_j; \quad r_j \in S \quad (*)$$

Let $\sum_{j \leq k} s_j y_j = 0$ be some homogeneous relation in $H^*(K/T)$ and assume that $s_k \neq 0$. Choose a sequence $i_1 \dots i_k$ such that $0 \neq A_{i_1} \dots A_{i_k}(s_k) \in \mathbb{F}$. Applying $A_{i_1} \dots A_{i_k}$ to the relation and using $(*)$ repeatedly we get a contradiction. \square

As an easy consequence of the above results, we recover the result of Kac [K]:

Theorem 6.7. *The Eilenberg-Moore spectral sequence for $K \xrightarrow{\pi} K/T \xrightarrow{\psi} BT$ collapses at E_2 . Furthermore, $H^*(K, \mathbb{F})$ is free over the image of π^* , and one has a short exact sequence of algebras:*

$$1 \rightarrow H^*(K/T, \mathbb{F}) \otimes_S \mathbb{F} \xrightarrow{\pi^*} H^*(K, \mathbb{F}) \longrightarrow \Lambda(x_1, \dots, x_r) \rightarrow 1.$$

Proof. Given a fibration $F \rightarrow E \rightarrow B$, with B simply connected, the Eilenberg-Moore spectral sequence is a second quadrant cohomological spectral sequence of graded algebras [JM]. The E_2 term is given by $E_2^{p,q} = \text{Tor}_{H^*(B)}^{-p,q}(\mathbb{F}, H^*(E))$, and it converges to $H^*(F, \mathbb{F})$ for a field \mathbb{F} .

We apply the spectral sequence to the fibration $K \xrightarrow{\pi} K/T \xrightarrow{\psi} BT$ in order to compute $H^*(K)$. The E_2 -term is given by:

$$E_2^{p,q} = \text{Tor}_{H^*(BT)}^{-p,q}(\mathbb{F}, H^*(K/T)).$$

From the previous theorem, we may write $H^*(K/T)$ as the S -module $S \otimes_{\mathbb{F}} H^*(K/T) \otimes_S \mathbb{F}$. It follows that

$$E_2^{p,q} = \text{Tor}_{H^*(BT)}^{-p,q}(\mathbb{F}, S) \otimes_{\mathbb{F}} H^*(K/T) \otimes_S \mathbb{F} = \Lambda(x_1, x_2, \dots, x_r) \otimes_{\mathbb{F}} H^*(K/T) \otimes_S \mathbb{F},$$

where $\Lambda(x_1, x_2, \dots, x_r)$ denotes an exterior algebra on classes x_i of homogeneous bidegree $(-1, |\sigma_i|)$. Due to degree reasons, this spectral sequence collapses. Consider the subring $H^*(K/T) \otimes_S \mathbb{F}$. Since it is in bidegree $(0, *)$, it can be identified with the edge homomorphism from $H^*(K/T)$ given by the image of π^* . This proves the above theorem. \square

Remark 6.8. *It is natural to ask if $H^*(K/T)$ and $H^*(K)$ are finitely generated algebras. If the characteristic of the field \mathbb{F} nonzero, then we will use results from the next section to show that both these algebras are not finitely generated, unless K is a compact Lie group.*

7. $H^*(K/T)$ AS A MODULE OVER THE STEENROD ALGEBRA \mathcal{A}_p

In this section, we study the structure of the cohomology of K and K/T as modules over the mod p Steenrod algebra \mathcal{A}_p .

Theorem 7.1. [K1] *Let p a prime and let $R = \mathbb{F}_p$. Let $\mathcal{P} = \sum P^i$ be the total Steenrod operation ($P^i = \text{Sq}^i$ if $p = 2$). Then $A_i(\mathcal{P}(x)) = (1 + \alpha_i^{p-1})\mathcal{P}(A_i(x))$ for any element $x \in H^*(K/T)$.*

Proof. First note that for $R = \mathbb{Z}$, we have the well defined formula:

$$A_i((h_i^*)^p) = \frac{(h_i^*)^p - (h_i^* - \alpha_i)^p}{\alpha_i} \equiv (-\alpha_i)^{p-1} \pmod{p}.$$

Applying the characteristic homomorphism ψ^* and reducing modulo p we get the formula: $A_i((\delta^{r_i})^p) = \alpha_i^{p-1}$. Let $x \in H^*(K/T)$. We may express it as $x = \delta^{r_i} \cup A_i(x) + z$. Then $\mathcal{P}(x) = (\delta^{r_i} + (\delta^{r_i})^p) \cup \mathcal{P}(A_i(x)) + \mathcal{P}(z)$. And thus $A_i(\mathcal{P}(x)) = (1 + \alpha_i^{p-1}) \cup \mathcal{P}(A_i(x))$ using the fact that A_i is a map of $H^*(K/K_i)$ -modules. \square

Now let $J = \langle \sigma_1, \sigma_2, \dots, \sigma_r \rangle$ be the ideal of generalized invariants. Let $2d_i$ denote the degree of the element σ_i . The degree of the top class in S is given by $2m = 2 \sum (d_i - 1)$. We will use the above theorem to show that $H^*(K/T)$ (and therefore $H^*(K)$) is locally finite as a module over \mathcal{A}_p (i.e. $\mathcal{A}_p(z)$ is a finite dimensional vector space for all $z \in H^*(K/T)$).

For any homogeneous subset $X \subseteq H^*(K/T)$ define $d(X) \leq \infty$ to be the highest degree of any homogeneous element in X . For $z \in H^*(K/T)$, let $M(z)$ be the S -module given by the span of elements of the form $s \cup a(z)$, $s \in S$, $a \in \mathcal{A}_p$. Note that $M(z)$ is a n \mathcal{A}_p -submodule of $H^*(K/T)$. Let $d(z)$ denote $d(M(z))$.

Theorem 7.2. *If z is an element of positive homogeneous degree $2k$, then $d(z) \leq 2k(m+1) - 2$.*

Proof. We work by induction on the degree of z . Since S is the subring of $H^*(K/T)$ generated by elements of degree 2, we are done for $k = 1$. Now let z be any element of homogeneous degree $2k + 2$. Let $x = \sum_{\mu} s_{\mu} \cup P^{\mu}(z)$ be a homogeneous element of $M(z)$ where μ ranges over finite sequences of positive integers $i_1 \dots i_s$ and $P^{\mu} = P^{i_1} \dots P^{i_s}$. By repeated application of the previous theorem we notice that $A_i(P^{\mu}(z)) \in M(A_i(z))$ for any $i \in I$. By induction, $P^{\mu}(z)$ can have degree at most $2k(m+1) - 2 + 2 = 2k(m+1)$. Thus x has degree at most $2k(m+1) + 2m$. Hence $d(z) = d(M(z)) \leq 2m + 2k(m+1) = 2(k+1)(m+1) - 2$ and we are done. \square

Corollary 7.3. *$H^*(K/T)$ and $H^*(K)$ are locally finite as modules over \mathcal{A}_p . In particular, over a field \mathbb{F} of positive characteristic $H^*(K/T)$ is finitely generated if and only if it is finite dimensional (i.e. if and only if K is a compact Lie group).*

Proof. The first part of the statement follows from the previous theorem. For the second part, notice that for $H^*(K/T)$ to be infinite dimensional and finitely generated, there must exist an element λ with arbitrary large nonzero powers. This is impossible since \mathcal{A}_p acts locally finitely on λ . The same argument works for $H^*(K)$. \square

8. THE HOPF ALGEBRA STRUCTURE OF $H^*(K)$

Recall the extension of algebras:

$$1 \rightarrow H^*(K/T) \otimes_S \mathbb{F} \xrightarrow{\pi^*} H^*(K) \longrightarrow \Lambda(x_1, \dots, x_r) \rightarrow 1.$$

We will show that this is actually an extension of Hopf algebras. It will be sufficient to show that $H^*(K/T) \otimes_S \mathbb{F}$ is a sub coalgebra of $H^*(K)$.

We define a coalgebra structure on $H^*(K/T)$ (introduced by D. Peterson) via:

$$\Delta(\delta^w) = \sum_{uv=w} \delta^u \otimes \delta^v$$

where the sum runs over all reduced expressions of w . Recall that $H^*(K/T) \otimes_S \mathbb{F}$ maps isomorphically to the image of π^* in $H^*(K)$. Our main theorem of this section will state that $\pi^* : H^*(K/T) \rightarrow H^*(K)$ is a map of coalgebras. The first step towards this goal is the construction of equivariant Schubert classes.

Let $H_T^*(K/T) = H^*(ET \times_T (K/T))$ denote the equivariant cohomology of K/T . For the moment, we allow coefficients in any ring. Define homomorphisms E_w for $w \in W$ by:

$$E_w : H_T^*(K/T) \xrightarrow{i_w^*} H_T^*(X_w) \xrightarrow{j_{[X_w]}} H_T^*(pt)$$

where $\int_{[X_w]}$ denotes integration over the fiber for the oriented fibration:

$$X_w \longrightarrow ET \times_T X_w \longrightarrow BT$$

Note that these are homomorphisms of $H_T^*(pt)$ -modules.

Claim 8.1. *There exists a unique basis $\{\delta_T^w\}_{w \in W}$ of $H_T^*(K/T)$ over $H_T^*(pt)$ with the property:*

$$E_v(\delta_T^w) = \begin{cases} 1 & \text{if } v = w \\ 0 & \text{otherwise} \end{cases}.$$

Proof. Uniqueness will follow easily once we have existence. We proceed by induction on $l(w)$. For $w = 1$, let $\delta_T^w = 1$. Assume we are done defining δ_T^w for $l(w) < k$. Using the naturality of the pullback:

$$\begin{array}{ccc} X_w & \longrightarrow & ET \times_T X_w \\ \downarrow & & \downarrow \\ pt & \longrightarrow & BT \end{array}$$

we see that δ_T^w restricts to δ^w under the restriction map $\iota^* : H_T^*(K/T) \rightarrow H^*(K/T)$.

For $w \in W$ such that $l(w) = k$, let $x_w \in H_T^{2k}(K/T)$ be any element that restricts to δ^w under the (surjective) map ι^* . It is easy to see that:

$$\begin{aligned} E_w(x_w) &= 1 \quad \text{and} \\ E_v(x_w) &= 0 \quad \text{if } l(v) \geq k, w \neq v \end{aligned}$$

Now δ_T^w can be defined as $\delta_T^w = x_w - \sum_{l(v) < k} E_v(x_w) \delta_T^v$, and we are done by induction. \square

One has a similar fibrations:

$$K/T \longrightarrow K \times_T (K/T) \xrightarrow{p} K/T, \quad X_w \longrightarrow K \times_T X_w \xrightarrow{p} K/T,$$

and we can define homomorphisms:

$$F_w : H^*(K \times_T (K/T)) \xrightarrow{i_w^*} H^*(K \times_T X_w) \xrightarrow{\int_{[X_w]}} H^*(K/T).$$

Note that F_w is a homomorphism of $H^*(K/T)$ -modules where $H^*(K \times_T (K/T))$ is viewed as a $H^*(K/T)$ -module via p^* .

Claim 8.2. *Let $\sigma^w = \Psi^*(\delta_T^w)$. Then the set $\{\sigma^w\}_{w \in W}$ is the unique basis of $H^*(K \times_T (K/T))$ as an $H^*(K/T)$ -module with the property:*

$$F_v(\sigma^w) = \begin{cases} 1 & \text{if } v = w \\ 0 & \text{otherwise} \end{cases}$$

Proof. Note that one has a pullback diagram:

$$\begin{array}{ccc} K \times_T (K/T) & \xrightarrow{\Psi} & ET \times_T (K/T) \\ \downarrow p & & \downarrow \\ K/T & \xrightarrow{\psi} & BT \end{array}$$

We may now use the previous claim and naturality with respect to pullback for the above diagram. \square

We will now need some technical results which will be the content of the next few lemmas. First consider the bundle over $\mathbb{C}P^1$ given by $K_i \rightarrow K_i/T$. Fix sections Y_i over the open cell $\mathbb{C} \subset \mathbb{C}P^1$ for each $i \in I$. Consider the following subspace of K :

$$Z_u = \bigcup_{u' \leq u} Y_{u'} \cdot T.$$

where Y_u is the space $Y_u = Y_{i_1} \times \dots \times Y_{i_k}$ with $u = r_{i_1} \dots r_{i_k}$ is a reduced expression. For $u, v \in W$, we can form the space $X_{u,v} = Z_u \times_T X_v$. The space $X_{u,v}$ has a decomposition:

$$X_{u,v} = \bigcup_{u' \leq u, v' \leq v} (Y_{u'} \cdot T) \times_T Y_{v'} = \bigcup_{u' \leq u, v' \leq v} Y_{u'} \times Y_{v'}.$$

Let $\mu : K \times_T (K/T) \rightarrow K/T$ denote the left action of K on K/T . One has an induced map:

$$\mu_{u,v} : X_{u,v} = Z_u \times_T X_v \longrightarrow K \times_T (K/T) \xrightarrow{\mu} K/T.$$

Let $\delta^{u,v} \in H^{2l(u)+2l(v)}(X_{u,v})$ be the class given by the cell: $Y_u \times Y_v$.

Lemma 8.3. *If $w \in W$ such that $l(w) = l(u) + l(v)$ then:*

$$\mu_{u,v}^*(\delta^w) = \begin{cases} \delta^{u,v} & \text{if } uv = w \\ 0 & \text{otherwise} \end{cases}$$

Proof. Recall from the CW-decomposition of K/T that we have a unique factorization of cells in K/T [Ku] given by: $Y_u \cdot Y_v = Y_{uv}$ if uv is reduced. Otherwise, $Y_u \cdot Y_v$ factors through cells of lower dimension. The result follows. \square

Now consider the oriented fibration

$$X_v \longrightarrow Z_u \times_T X_v \xrightarrow{p_{u,v}} X_u$$

Lemma 8.4.

$$\int_{[X_v]} \delta^{u,v} = \delta^u$$

Proof. The proof is obvious since the Serre spectral sequence for the above filtration collapses at E_2 and $\delta^{u,v}$ is represented by $\delta^u \otimes \delta^v$. \square

Recall the action map

$$\mu : K \times_T (K/T) \longrightarrow K/T$$

Lemma 8.5.

$$\mu^*(\delta^w) = \sum_{uv=w} \delta^u \cup \sigma^v$$

where the sums runs over all reduced expressions of w , and δ^u is identified with $p^*(\delta^u)$.

Proof. Let $l(w) = k$. We can write

$$\mu^*(\delta^w) = \sum_{l(u)+l(v)=k} a_{u,v} \delta^u \cup \sigma^v$$

where $a_{u,v}$ are elements in the coefficient ring. We use the operators F_v to isolate them:

$$F_v \mu^*(\delta^w) = \sum_{l(u)=k-l(v)} a_{u,v} \delta^u$$

and therefore $i_u^* F_v \mu^*(\delta^w) = a_{u,v} \delta^u$ where $i_u : X_u \rightarrow K/T$ is the inclusion. Now consider the commutative diagram:

$$\begin{array}{ccccc} Z_u \times_T X_v & \longrightarrow & K \times_T X_v & \xrightarrow{i_v} & K \times_T (K/T) \\ \downarrow \mu_{u,v} & & \downarrow \mu & & \downarrow \mu \\ X_{uv} & \longrightarrow & K/T & \xrightarrow{=} & K/T \end{array}$$

Thus

$$i_u^* F_v \mu^*(\delta^w) = \int_{[X_v]} \mu_{u,v}^*(\delta^w) = \begin{cases} \delta^u & \text{if } uv = w \\ 0 & \text{otherwise} \end{cases}$$

using 8.3 and 8.4, so we are done. \square

Let $m : K \times K \rightarrow K$ denote the multiplication map. Consider the commutative diagram:

$$\begin{array}{ccc} K \times K & \xrightarrow{\tilde{\pi}} & K \times_T (K/T) \\ \downarrow m & & \downarrow \mu \\ K & \xrightarrow{\pi} & K/T \end{array}$$

Let π_1, π_2 be defined as the projections onto the first and second factor respectively:

Lemma 8.6. *In cohomology, $\tilde{\pi}$ is given by*

- a) $\tilde{\pi}^*(\sigma^v) = \pi_2^* \circ \pi^*(\delta^v)$
- b) $\tilde{\pi}^*(\delta^u) = \pi_1^* \circ \pi^*(\delta^u)$

Proof. For a), recall that by definition $\sigma^v = \Psi^*(\delta_T^v)$. Now notice that we have a commutative diagram:

$$\begin{array}{ccc} K \times K & \xrightarrow{f} & ET \times (K/T) \\ \downarrow \tilde{\pi} & & \downarrow g \\ K \times_T (K/T) & \xrightarrow{\Psi} & ET \times_T (K/T) \end{array}$$

Note that $ET \times (K/T) \xrightarrow{\sim} K/T$ since ET is contractible. Under this identification $f = \pi \circ \pi_2$ and g is the inclusion of K/T in $ET \times_T (K/T)$. Since $g^*(\delta_T^v) = \delta^v$, we get:

$$\tilde{\pi}^*(\sigma^v) = \tilde{\pi}^* \circ \Psi^*(\delta_T^v) = \pi_2^* \circ \pi^*(\delta^v).$$

Part b) follows from the commutative diagram:

$$\begin{array}{ccc} K \times K & \xrightarrow{\tilde{\pi}} & K \times_T (K/T) \\ \downarrow \pi_1 & & \downarrow p \\ K & \xrightarrow{\pi} & K/T. \end{array}$$

\square

We now prove the main theorem of this section:

Theorem 8.7. *The map $\pi^* : H^*(K/T) \rightarrow H^*(K)$ is a map of coalgebras, where cohomology is taken with coefficient in a field.*

Proof. One recalls the commutative diagram:

$$\begin{array}{ccc} K \times K & \xrightarrow{\tilde{\pi}} & K \times_T (K/T) \\ \downarrow m & & \downarrow \mu \\ K & \xrightarrow{\pi} & K/T \end{array}$$

Now one invokes 8.5 and 8.6, to get the required equality:

$$m^* \pi^*(\delta^w) = \sum_{uv=w} \pi_1^*(\pi^*(\delta^u)) \cup \pi_2^*(\pi^*(\delta^v)),$$

where the sum is begin taken over all reduced expressions. This is exactly the statement of the theorem. \square

Remark 8.8. *Note that 8.3, 8.4, 8.5 and 8.6 made no assumption on the coefficient ring. For an arbitrary ring R , define $A_R = \text{Im}(\pi^* H^*(K/T; R) \rightarrow H^*(K; R))$. Note that for a field \mathbb{F} , we have shown that $A_{\mathbb{F}} = H^*(K/T; \mathbb{F}) \otimes_S \mathbb{F}$. It follows easily that [K1]:*

- 1) $A_{\mathbb{F}} = A_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{F}$,
- 2) $A_{\mathbb{Z}}$ has the structure of a Hopf algebra.

Thus $A_{\mathbb{Z}}$ is an integral lift of the Hopf algebras $A_{\mathbb{F}}$.

9. ON THE RING STRUCTURE OF $H_*(K)$

We fix a coefficient field \mathbb{F} of positive characteristic throughout this section. It is our aim to show that the Pontrjagin ring: $H_*(K) = H_*(K, \mathbb{F})$ is a finitely generated algebra. We begin with some preliminary lemmas:

Lemma 9.1. *The left action of $H_*(K)$ on $H_*(K/T)$ factors through the projection $H_*(K) \rightarrow A_{\mathbb{F}}^*$. Moreover, $H_*(K/T)$ is a finitely generated free left $A_{\mathbb{F}}^*$ -module.*

Proof. The first part of the above lemma is easy to see and we leave it to the reader. The theory of Hopf-algebras can now be used to show that $H_*(K/T)$ is a free $A_{\mathbb{F}}^*$ -module [MM](Thm. 4.4). An easy argument using Poincaré series shows that a basis set generates the finite dimensional vector space S^* , which proves the lemma.

For the benefit of the reader, we also provide an alternate proof: Recall that for a topological group G , and a principal G -bundle $G \rightarrow E \rightarrow B$, one has a natural homological (Bar) spectral sequence of coalgebras converging to $H_*(B)$ with E_2 -term given by $E_2^{p,q} = \text{Tor}_{H_*(G)}(\mathbb{F}, H_*(E))$ [JM]. Consider a pair of pullbacks of principal T -bundles:

$$\begin{array}{ccccc} K \times K & \xrightarrow{m} & K & \longrightarrow & ET \\ \downarrow Id \times \pi & & \downarrow \pi & & \downarrow \\ K \times (K/T) & \xrightarrow{\mu} & K/T & \xrightarrow{\psi} & BT \end{array}$$

This induces an action of $H_*(K)$ on the Bar spectral sequence of converging to $H_*(K/T)$:

$$H_*(K) \otimes \text{Tor}_{H_*(T)}(\mathbb{F}, H_*(K)) \longrightarrow \text{Tor}_{H_*(T)}(\mathbb{F}, H_*(K)).$$

Notice that the homogeneous space K/T does not change if we replace K by its semisimple factor, and T by the summand generated by the coroot spaces. Hence we may assume that K is simply connected, and that the inclusion of the maximal torus $T \subset K$ is null homotopic. Consequently, we have:

$$\mathrm{Tor}_{H_*(T)}(\mathbb{F}, H_*(K)) = H_*(K) \otimes_{\mathbb{F}} \mathrm{Tor}_{H_*(T)}(\mathbb{F}, \mathbb{F}).$$

Differentials in this spectral sequence must annihilate the piece corresponding to the dual exterior algebra: $\Lambda^*(x_1, \dots, x_r) \subseteq H_*(K)$. This dual algebra is itself an exterior algebra. The generators of this exterior algebra must therefore be targets of differentials originating on elements indecomposable under the $H_*(K)$ -action. Now we may write:

$$\mathrm{Tor}_{H_*(T)}(\mathbb{F}, \mathbb{F}) = \Gamma(y_1, \dots, y_r) = \mathbb{S}^* \otimes_{\mathbb{F}} \Gamma(\tau_1, \dots, \tau_r)$$

where $\Gamma(y_1, \dots, y_r)$ denotes the dual of a polynomial algebra. This coalgebra is bigraded by giving y_i bidegree $(1, 1)$. The element τ_i is an element of bidegree $(1, |x_i|)$, the vector space \mathbb{S}^* is dual to \mathbb{S} , and is detected in $H_*(BT)$. It follows that the generators τ_i must hit a set of generators of the dual exterior algebra $\Lambda^*(x_1, \dots, x_r)$ in the spectral sequence. Consequently, the E_∞ term of the spectral sequence is a free left module over $A_{\mathbb{F}}^* = H_*(K) \otimes_{\Lambda^*(x_1, \dots, x_r)} \mathbb{F}$, with a basis given by the finite dimensional vector space \mathbb{S}^* . The result follows from an easy filtration argument. \square

We now need the following general lemma:

Lemma 9.2. *Let A be a (not necessarily commutative) finitely generated, graded, connected \mathbb{F} -algebra. Let $B \subseteq A$ be a graded sub algebra so that A is finitely generated as a left B -module. Then B is also a finitely generated algebra.*

Proof. Let $\{a_1, \dots, a_n\}$ be a set of algebra generators of A over \mathbb{F} , and let $\{c_1, \dots, c_m\}$ be a basis set of A as a left B -module, with $c_1 = 1$. We pick a finite set $\{e_1, \dots, e_k\} \subseteq B$, so that:

$$a_j \in \sum_{r,s} \mathbb{F} e_r c_s, \quad c_i a_j \in \sum_{r,s} \mathbb{F} e_r c_s, \quad j \leq n, i \leq m.$$

Let $b \in B$ be an arbitrary element. Since A is finitely generated, there exists a polynomial f so that $f(a_1, \dots, a_n) = b$. Using the above properties repeatedly, we may write

$$b = \sum_i g_i(e_1, \dots, e_k) c_i,$$

for some polynomials g_i . But since A is a free left B -module, we observe that $g_i = 0$ for all $i > 1$ and that $b = g_1(e_1, \dots, e_k)$. It follows that B is generating by $\{e_1, \dots, e_k\}$. \square

As an easy consequence of the above lemmas, we have:

Theorem 9.3. *Let \mathbb{F} be a field of positive characteristic. Then the dual hopf algebra $A_{\mathbb{F}}^*$ is a finitely generated \mathbb{F} -algebra. It follows that $H_*(K, \mathbb{F})$ is also finitely generated \mathbb{F} -algebra.*

Proof. Dualizing the coalgebra structure of $H^*(K/T)$, we observe that $H_*(K/T)$ has the structure of a finitely generated algebra on the set of elements δ_{r_i} dual to the Schubert basis elements δ^{r_i} . Working with coefficients in a field \mathbb{F} of positive characteristic, the results of previous sections shows that $A_{\mathbb{F}}^* \subseteq H_*(K/T)$ is a sub algebra. By the first lemma, we see that $H_*(K/T)$ is a finitely generated, free, left $A_{\mathbb{F}}^*$ -module and so the second lemma implies that $A_{\mathbb{F}}^*$ is a finitely generated algebra. The result about $H_*(K, \mathbb{F})$ follows easily once we describe it as an extension of $A_{\mathbb{F}}^*$ by $\Lambda^*(x_1, \dots, x_r)$. \square

10. EXAMPLES OF RANK TWO

In this section, we describe the structure of the (co)homology of rank two Kac-Moody groups and their flag varieties. By a rank two Kac-Moody group, we shall mean a Kac-Moody group for which the set I has cardinality two.

Generalized Cartan matrices representing Kac-Moody groups of rank two are given by:

$$A(a, b) = \begin{pmatrix} 2 & -a \\ -b & 2 \end{pmatrix}.$$

Throughout this section let $K = K(a, b)$ denote the semisimple factor inside the corresponding unitary form. If $ab < 4$, then K is a compact Lie group. In particular

$$\begin{aligned} (a, b) = (0, 0) & \quad K = SU(2) \times SU(2) \\ (a, b) = (2, 1) & \quad K = \text{Spin}(5) = Sp(2) \\ (a, b) = (1, 1) & \quad K = SU(3) \\ (a, b) = (1, 3) & \quad K = G_2 \end{aligned}$$

Henceforth, we only work with a generalized Cartan matrix $A = A(a, b)$ with $ab \geq 4$. Let $T \subset K$ denote the maximal torus. Then the Weyl group has a presentation given by: $W = \langle r_1, r_2 \mid r_1^2 = r_2^2 = 1 \rangle$. Thus the Poincaré series for $H^*(K/T; \mathbb{Z})$ is

$$P_t H^*(K/T; \mathbb{Z}) = 1 + 2t^2 + 2t^4 + \dots = \frac{1 + t^2}{1 - t^2}$$

Hence $H^*(K/T, \mathbb{Z})$ contains two elements from the Schubert basis in every positive even degree. Let δ_n be the element δ^w where $l(w) = n$, $l(wr_1) < l(w)$. Thus $w = \dots r_1 r_2 r_1$ (n terms). Let τ_n be the other element from the Schubert basis in the same degree. Denote δ_1 and τ_1 by δ and τ respectively. The action of the Weyl group on τ and δ can be easily deduced from Claim 5.2.

Given a generalized Cartan matrix $A = A(a, b)$, define integers c_i, d_i recursively via:

$$c_0 = d_0 = 0; \quad c_1 = d_1 = 1; \quad c_{j+1} = ad_j - c_{j-1}; \quad d_{j+1} = bc_j - d_{j-1}.$$

Theorem 10.1. *In $H^*(K/T; \mathbb{Z})$ we have the relations:*

$$\begin{aligned} \delta \cup \delta_n &= d_{n+1} \delta_{n+1}; & \delta \cup \tau_n &= \delta_{n+1} + d_n \tau_{n+1} \\ \tau \cup \tau_n &= c_{n+1} \tau_{n+1}; & \tau \cup \delta_n &= \tau_{n+1} + c_n \delta_{n+1} \end{aligned}$$

Proof. We proceed by induction. Since $\delta_0 = \tau_0 = 1$, the result is true for $n = 0$. Now write $\delta \cup \delta_n = A\delta_{n+1} + B\tau_{n+1}$ where $A, B \in \mathbb{Z}$. We recall the annihilation operators A_1 and A_2 acting diagonally with respect to the Schubert basis. Note that $A_1(\delta \cup \delta_n) = A_1(A\delta_{n+1} + B\tau_{n+1}) = A\tau_n$, but on the other hand we have the twisted derivation property given by theorem 5.7 and claim 5.2:

$$\begin{aligned} A_1(\delta \cup \delta_n) &= r_1(\delta) \cup A_1(\delta_n) + \delta_n \cup A_1(\delta) \\ &= (\delta - (2\delta - b\tau)) \cup \tau_{n-1} + \delta_n \\ &= b\tau \cup \tau_{n-1} - \delta \cup \tau_{n-1} + \delta_n \\ &= bc_n \tau_n - \delta_n - d_{n-1} \tau_n + \delta_n \\ &= d_{n+1} \tau_n \end{aligned}$$

where we used induction and the recursive definition of d_i . Thus $A = d_{n+1}$. Now we apply A_2 and observe that $B = 0$. The other equalities follow similarly. \square

Definition 10.2. Given a generalized cartan matrix $A(a, b)$, we define the generalized binomial coefficients ¹:

$$D(n, m) = \frac{d_{n+m} d_{n+m-1} \cdots 1}{d_n d_{n-1} \cdots 1 \, d_m d_{m-1} \cdots 1} \quad \text{and}$$

$$C(n, m) = \frac{c_{n+m} c_{n+m-1} \cdots 1}{c_n c_{n-1} \cdots 1 \, c_m c_{m-1} \cdots 1}$$

Note that if $a = b = 2$, then $c_n = d_n = n$ and thus $C(n, m) = D(n, m) = \binom{n+m}{n}$.

The previous theorem on the ring structure of $H^*(K/T, \mathbb{Z})$ immediately implies the following theorem about the cohomology of the partial flag varieties:

Theorem 10.3. Let K_1, K_2 be the maximal compact subgroups of the standard parabolic subgroups corresponding to $\{1\}, \{2\} \subset \{1, 2\}$ respectively. Then

$$H^{2n}(K/K_1; \mathbb{Z}) = \mathbb{Z}\langle \tau_n \rangle; \quad \tau_n \cup \tau_m = C(n, m) \tau_{n+m}$$

$$H^{2n}(K/K_2; \mathbb{Z}) = \mathbb{Z}\langle \delta_n \rangle; \quad \delta_n \cup \delta_m = D(n, m) \delta_{n+m}$$

In particular we see that the generalized binomial coefficients $C(n, m)$ and $D(n, m)$ are integers! ²

Remark 10.4. The above theorems completely determine the ring structure of $H^*(K/T; \mathbb{Z})$, and $H^*(K/K_i, \mathbb{Z})$.

Claim 10.5. $H^*(K/T; \mathbb{Q}) = \mathbb{Q}[\delta, \tau]/J$, where J is the ideal given by the quadratic relation: $a\delta^2 + b\tau^2 - ab\delta\tau = 0$. In particular $H^*(K/T; \mathbb{Q})$ is generated by $H^2(K/T; \mathbb{Q})$.

Proof. In $H^4(K/T; \mathbb{Z})$ we have a relation $a\delta^2 + b\tau^2 - ab\delta\tau = 0$. This yields a map:

$$\mathbb{Q}[\delta, \tau]/J \longrightarrow H^*(K/T; \mathbb{Q}).$$

Using the ring structure of $H^*(K/T, \mathbb{Q})$, this map is surjective. To see that it is an isomorphism, one simply compares the Poincaré series. We leave this as an exercise. \square

The additive structure of $H^*(K, \mathbb{Z})$, and the Hopf algebras $A_{\mathbb{Z}}, A_{\mathbb{F}_p}$:

Now consider the Serre spectral sequence in integral cohomology for the fibration:

$$T \longrightarrow K \longrightarrow K/T.$$

Let $H^1(T, \mathbb{Z}) = \mathbb{Z}\alpha \oplus \mathbb{Z}\beta$, with $d_2(\alpha) = \delta$ and $d_2(\beta) = \tau$. The ring structure of $H^*(K/T, \mathbb{Z})$ now allows us to compute the structure of the differential d_2 . It is easy to see that d_2 is injective on $E_2^{*,2}$, and that $E_3 = E_\infty$. Let $g_n = \gcd(c_n, d_n)$ denote the g.c.d of the pair c_n, d_n . The following results are easy consequences of the explicit formulas given in theorem 10.1:

$$E_3^{2n,0} \cong E_3^{2n+2,1} \cong \mathbb{Z}/g_n\mathbb{Z}.$$

Corollary 10.6. The additive structure of $H^*(K, \mathbb{Z})$ is given by:

$$H^{2n+3}(K, \mathbb{Z}) = H^{2n}(K, \mathbb{Z}) = \mathbb{Z}/g_n\mathbb{Z}.$$

¹This terminology is due to Haynes Miller

²We thank Kasper Andersen for showing us a nice algebraic proof of integrality.

From the above Serre spectral sequence, we notice that $E_3^{*,0}$ is given by the Hopf algebra $A_{\mathbb{Z}} = \text{Im}(H^*(K/T, \mathbb{Z}) \rightarrow H^*(K, \mathbb{Z}))$, and it can be identified with $H^*(K/T, \mathbb{Z}) \otimes_S \mathbb{Z}$. Recall that in degree $2n$, $A_{\mathbb{Z}}$ is a cyclic group of order g_n generated by δ_n or τ_n :

$$A_{\mathbb{Z}}^{2n} = \mathbb{Z}/g_n\mathbb{Z}; \quad g_n = \text{gcd}(c_n, d_n)$$

The coalgebra structure on $A_{\mathbb{Z}}$ was induced by:

$$\Delta(\delta^w) = \sum_{uv=w} \delta^u \otimes \delta^v$$

where the sum is over all reduced expressions, thus

$$\begin{aligned} \Delta(\delta_n) &= \sum_{i=0}^n \delta_i \otimes \delta'_{n-i}; & \delta'_{n-i} &= \begin{cases} \delta_{n-i} & \text{if } i \text{ even} \\ \tau_{n-i} & \text{if } i \text{ odd} \end{cases} \\ \Delta(\tau_n) &= \sum_{i=0}^n \tau_i \otimes \tau'_{n-i}; & \tau'_{n-i} &= \begin{cases} \tau_{n-i} & \text{if } i \text{ even} \\ \delta_{n-i} & \text{if } i \text{ odd} \end{cases} \end{aligned}$$

Now fix a prime p . Recall that $A_{\mathbb{F}_p} = \mathbb{F}_p \otimes_{\mathbb{Z}} A_{\mathbb{Z}}$. Hence to understand $A_{\mathbb{F}_p}$, we need to know when p divides g_n . We have the following theorem on the arithmetic properties of the integers c_n and d_n :

Theorem 10.7. *Let $g_n = \text{gcd}(c_n, d_n)$. Given a prime p , there is a smallest positive integer k with the property that p divides g_k . Further, p divides g_n if and only if k divides n . More precisely, k is given by:*

- 1) $k = 2p$ if p divides a or b but not both
- 2) $k = p$ if $ab \equiv 4 \pmod{p}$ and the conditions of 1) do not hold. In all other cases, we have:
- 3) $k = r$ where r is the multiplicative order of any root of the quadratic polynomial given by $x^2 - (ab - 2)x + 1$ defined over $\mathbb{F}_{p^2}[x]$.

Proof. Note that the c_i, d_i and g_i have terms that look like:

i	c_i	d_i	g_i
0	0	0	0
1	1	1	1
2	a	b	(a, b)
3	$ab - 1$	$ab - 1$	$ab - 1$
4	$a(ab - 2)$	$b(ab - 2)$	$(a, b)(ab - 2)$

So the claim can be verified for $p = 2$ explicitly. Assuming p is odd, consider the generating function

$$F(x) = \sum_{i=0}^{\infty} \begin{pmatrix} c_i \\ d_i \end{pmatrix} x^i$$

This generating function is to be thought of as a formal power series with coefficients in the two dimensional vector space over the field \mathbb{F}_p . Thus we are interested in when the coefficient of x^n is zero. Now, we have a functional equation:

$$\left(x^2 - \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} x + I \right) F(x) = x \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

i.e.

$$(x - M)(x - M^{-1})F(x) = x \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

where $M = \frac{1}{2} \begin{pmatrix} \mu & a \\ b & \mu \end{pmatrix}$ and $\mu = \sqrt{ab - 4} \in \mathbb{F}_{p^2}$. Thus

$$(8) \quad F(x) = \frac{1}{(x - M)(x - M^{-1})} x \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

First consider the case $ab \equiv 4 \pmod{p}$. In this case $M = M^{-1}$, so (8) says

$$F(x) = \sum_{i=0}^{\infty} ix^i M^{i+1} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Since M is invertible, the coefficients of x^n are zero if and only if n is a multiple of p , and that is what we wanted to show. For all other cases $M - M^{-1}$ is invertible, so (8) becomes:

$$F(x) = \sum_{i=0}^{\infty} x^i \begin{pmatrix} M^i - M^{-i} \\ M - M^{-1} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Thus we are interested in n where

$$\begin{pmatrix} M^n - M^{-n} \\ M - M^{-1} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

i.e.

$$M^{2n} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Now consider the case when p divides a or b but not both. Assume without loss of generality $p|b$. Then $M = \begin{pmatrix} \eta & \frac{a}{2} \\ 0 & \eta \end{pmatrix}$, and $\eta = \sqrt{-1} \in \mathbb{F}_{p^2}$. Hence

$$M^{2n} = \begin{pmatrix} (-1)^n & (-1)^{n+1} na\eta \\ 0 & (-1)^n \end{pmatrix}$$

and $M^{2n} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ if and only if n is a multiple of $2p$.

In all remaining cases, M is diagonalizable over \mathbb{F}_{p^2} . Since M^2 has determinant 1, the only way the equality

$$M^{2n} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

can hold is that M^{2n} is the identity matrix. This is equivalent to the condition that the eigenvalues of M^2 have multiplicative order dividing n . These eigenvalues are exactly the roots of the characteristic polynomial of M^2 , which is given by $x^2 - (ab - 2)x + 1$. \square

From the above theorem, we observe that the Poincaré series if $A_{\mathbb{F}_p}$ is:

$$P_t(A_{\mathbb{F}_p}) = 1 + t^{2k} + t^{4k} + \dots = \frac{1}{1 - t^{2k}}.$$

and both δ_{mk}, τ_{mk} are generators in that degree. It is easier to understand the dual $A_{\mathbb{F}_p}^*$, which turns out to be a polynomial algebra:

Claim 10.8. $A_{\mathbb{F}_p}^* = \mathbb{F}_p[x_{2k}]$ where x_{2k} is a primitive class in degree $2k$.

Proof. Let $x \in A_{\mathbb{F}_p}^*$ be any generator in degree $2k$. We prove by induction that x^n is a generator in degree $2nk$. Let $\tau_n \in A_{\mathbb{F}_p}$ be some generator in degree $2nk$. We have:

$$\Delta(\tau_n) = \sum_{i=0}^n \lambda_i \tau_i \otimes \tau_{n-i}, \quad \lambda_i \neq 0$$

Now using the induction hypothesis, we get:

$$\langle x \cdot x^{n-1}, \tau_n \rangle = \langle x \otimes x^{n-1}, \Delta\tau_n \rangle = \lambda_1 \langle x \otimes x^{n-1}, \tau_1 \otimes \tau_{n-1} \rangle \neq 0$$

Thus x^n generates $A_{\mathbb{F}_p}^*$ in degree $2nk$. □

Theorem 10.9. *There is an isomorphism of algebras:*

$$H_*(K; \mathbb{F}_p) = \Lambda(y_3, y_{2k-1}) \otimes \mathbb{F}_p[x_{2k}]$$

where the subscripts denote the homogeneous degree of the generators. These generators are related via a higher Bockstein homomorphism: $\beta^{(m)}x_{2k} = y_{2k-1}$, m being the exponent of p in g_k . Moreover, the generators y_3, y_{2k-1} are primitive, and if p is odd, then so is the generator x_{2k} .

Proof. Recall the short exact sequence of Hopf algebras:

$$1 \rightarrow A_{\mathbb{F}_p} \longrightarrow H^*(K; \mathbb{F}_p) \longrightarrow \Lambda(x_1, x_2) \rightarrow 1$$

On dualizing, we get:

$$1 \rightarrow \Lambda(z_1, z_2) \longrightarrow H_*(K; \mathbb{F}_p) \longrightarrow \mathbb{F}_p[x_{2k}] \rightarrow 1$$

Now recall that $H_*(K; \mathbb{Q}) = \Lambda(z)$, $|z| = 3$. This forces $|z_1| = 3$, where z_1 is a permanent cycle in the Bockstein spectral sequence for $H_*(K; \mathbb{F}_p)$ and $|z_2| = 2k - 1$, where z_2 is the target of a higher Bockstein of height m supported on x_{2k} . Let us relabel these classes by their subscript and call them y_3 and y_{2k-1} respectively. It is clear for dimensional reasons that these classes are primitive. Now since $\mathbb{F}_p[x_{2k}]$ is a free algebra, we may fix a section to the above short exact sequence. Again, for dimensional reasons, the class x_{2k} is primitive, with the possible exception of the case when $p = 2, k = 3$ and the coproduct on x_{2k} is given by:

$$\Delta(x_6) = 1 \otimes x_6 + x_6 \otimes 1 + x_3 \otimes x_3.$$

Finally, to show that $H_*(K; \mathbb{F}_p)$ is a tensor product of $\mathbb{F}_p[x_{2k}]$ and $\Lambda(x_1, x_2)$, it is sufficient to show that $[x_{2k}, y_3] = [x_{2k}, y_{2k-1}] = 0$. This is easy to establish since both the elements: $[x_{2k}, y_3]$ and $[x_{2k}, y_{2k-1}]$ are primitive, but on the other hand, there are no non-zero primitive elements in those degrees. □

For the sake of completeness, we include the following theorem.

Theorem 10.10. *Given an odd prime p , let $\nu_p(s)$ denote the exponent of the prime p dividing s . Let g_k denote the first integer in the sequence $\{g_n\}$, so that p divides g_k . Then we have:*

$$\nu_p(g_{sk}) = \nu_p(s) + \nu_p(g_k).$$

Proof. Our proof of the above theorem is not arithmetic. We know from the previous theorem that $H^*(K, \mathbb{F}_p)$ supports a higher Bockstein homomorphism of height $\nu_p(g_k)$. Now the theory of torsion in H-spaces [Ka](12.1, Thm A) shows that the Bockstein spectral sequence is forced once we know the first differential. This translates to the statement of our theorem. \square

11. THE STABLE TRANSFER FROM BK TO $BN(T)$

It is well known for a compact Lie group G , with maximal torus T and normalizer N , that the suspension spectrum of BG_+ is a stable retract of BN_+ . The retraction is constructed as a transfer map, and uses the essential fact that G/N is a finite complex (with Euler characteristic equal to one). In the case of Kac-Moody groups K , the space $K/N(T)$ is not even homologically finite, and so there is no (a priori) obvious transfer map. Nevertheless, in this section we will construct a transfer using the homotopy decomposition of the spaces BK and $BN(T)$ given in [Ki, Ki2, BK]. The author would like to thank Bill Dwyer for motivating the argument used in this section. The construction of the stable transfer proceeds as follows:

Let \mathbf{BK}_+ and $\mathbf{BN}(\mathbf{T})_+$ denote the suspension spectra of BK_+ and $BN(T)_+$, respectively each endowed with a disjoint base point. In order to construct a stable transfer map from \mathbf{BK}_+ to $\mathbf{BN}(\mathbf{T})_+$, first recall [Ki, Ki2, BK] that the following canonical maps are homotopy equivalences:

$$\mathrm{hocolim}_{J \in \mathcal{S}(A)} BN_J(T) \longrightarrow BN(T), \quad \mathrm{hocolim}_{J \in \mathcal{S}(A)} BK_J \longrightarrow BK$$

Let us fix a representation \mathcal{V} of the Kac-Moody group K with a countable basis, and the property that given $J \in \mathcal{S}(A)$, every representation of K_J appears in \mathcal{V} with infinite multiplicity. Such a representation is easy to construct: for example, we may take countable sums of all representations of K of the form $L_\mu \otimes L_\tau$, where μ is a dominant weight, τ is an anti-dominant weight, and L_μ (resp. L_τ) denote the highest (resp. lowest) weight irreducible representations of K .

Let Met denote contravariant functor on the category $\mathcal{S}(A)$, taking values in spaces given by $\mathrm{Met}(J) = \mathrm{Met}_{K_J}(\mathcal{V})$: the contractible space of K_J -invariant metrics on \mathcal{V} . An easy spectral sequence argument shows that the homotopy inverse limit of this functor is weakly contractible and so, in particular, it is non-empty. An element in this homotopy inverse limit may be interpreted as a family of metrics parametrized over the simplicial complex $|\mathcal{S}(A)|$ (given by the geometric nerve of $\mathcal{S}(A)$) and with the property that the metrics over the face corresponding to the sequence of inclusions: $J_0 < \dots < J_k$ are K_{J_0} -invariant. Fixing such an element describes \mathcal{V} as a complete universe parametrized over $|\mathcal{S}(A)|$. Working with this universe throughout, let \mathbf{S}^0 denote the equivariant sphere spectrum. In addition, let $\mathbf{K}_J/\mathbf{N}_J(\mathbf{T})_+$ denote the suspension spectrum of the K_J -space $K_J/N_J(T)$, endowed with a disjoint base point.

The subtle part in the construction of the transfer will be to construct a zig-zag of spectra:

$$\mathbf{T} : \mathrm{hocolim}_{J \in \mathcal{S}(A)} \mathbf{S}^0 \xleftarrow{\sim} \mathbf{X} \longrightarrow \mathrm{hocolim}_{J \in \mathcal{S}(A)} \mathbf{K}_J/\mathbf{N}_J(\mathbf{T})_+.$$

Each spectrum above will be parametrized over the simplicial complex $|\mathcal{S}(A)|$. The maps in \mathbf{T} will fiber over self maps of $|\mathcal{S}(A)|$ which preserves the faces (though not pointwise). By construction, each spectrum above will admit the fiberwise action of the group K_{J_0} over

the face of $|\mathcal{S}(A)|$ corresponding to the sequence of inclusions $J_0 < \dots < J_k$. Moreover, \mathbf{T} will be equivariant with respect to K_{J_0} , over this face. If we let $\mathbf{T}(J)$ denote the map over the vertex of $|\mathcal{S}(A)|$ given by the object $J \in \mathcal{S}(A)$, then our construction will also show that $\mathbf{T}(J)$ is equivalent to the standard equivariant splitting of \mathbf{S}^0 from $\mathbf{K}_J/\mathbf{N}_J(\mathbf{T})_+$.

Taking homotopy orbits of \mathbf{T} , and inverting the equivalence in the zig-zag, we get a map which we will define as the stable transfer $\overline{\mathbf{T}} : \mathbf{BK}_+ \rightarrow \mathbf{BN}(\mathbf{T})_+$:

$$\overline{\mathbf{T}} : \text{hocolim}_J \mathbf{EK}_+ \wedge_{\mathbf{K}_J} \mathbf{S}^0 \longrightarrow \text{hocolim}_J \mathbf{EK}_+ \wedge_{\mathbf{K}_J} (\mathbf{K}_J/\mathbf{N}_J(\mathbf{T})_+).$$

So it remains to actually construct a map \mathbf{T} with all the required properties. We begin with some auxiliary constructions. Define \mathbf{X} as the parametrized spectrum given by the co-end construction induced by obvious restrictions:

$$\mathbf{X} = \coprod_{J_0 < \dots < J_k} \Delta^k \times (\text{Emb}(K_{J_k}(A)/N_{J_k}(T))_+ \wedge \mathbf{S}^0) / \sim,$$

where $\text{Emb}(K_J/N_J(T))$ is the space of K_J -equivariant embeddings of $K_J/N_J(T)$ in \mathcal{V} . The space $\text{Emb}(K_J/N_J(T))$ is contractible, and so it is clear that the parametrized projection map from \mathbf{X} to the parametrized spectrum: $\text{hocolim}_{J \in \mathcal{S}(A)} \mathbf{S}^0$, is an equivalence.

The construction of our stable transfer \mathbf{T} reduces to the construction of a face-preserving map from \mathbf{X} to the spectrum $\text{hocolim}_{J \in \mathcal{S}(A)} \mathbf{K}_J/\mathbf{N}_J(\mathbf{T})_+$, as parametrized family of equivariant splitting maps constructed using the Pontrjagin-Thom collapse construction for equivariant embeddings of $K_J/N_J(T)$ in \mathcal{V} . The subtlety here is to ensure coherence between the individual maps. To address the coherence problem, we will take advantage of the following two general facts (which are easy to prove, and are left to the reader):

- (1) Given $J \in \mathcal{S}(A)$, let \mathfrak{h}_J denote the Lie algebra: $\mathfrak{h}_J = \langle h \in \mathfrak{h}, \mid \alpha_j(h) = 0, j \in J \rangle$. Then the infinitesimal action of \mathfrak{h}_J on $K_S/N_S(T)$ is K_J -invariant for any $J < S$. Moreover, the \mathfrak{h}_J -fixed set of $K_S/N_S(T)$ is given by $K_J/N_J(T)$.
- (2) Fix elements $\rho_J \in \mathfrak{h}_J$ with the property $\alpha_i(\rho_J) = 1$ for $i \notin J$. Then the simplicial complex $|\mathcal{S}(A)|$ given by the geometric nerve of $\mathcal{S}(A)$ can be canonically identified with an affine subspace of \mathfrak{h} , determined by the property that the vertex of $|\mathcal{S}(A)|$ corresponding to $J \in \mathcal{S}(A)$, maps to the element ρ_J .

Now given an embedding $e : K_J/N_J(T) \rightarrow \mathcal{V}$, recall that the Pontrjagin-Thom construction is given by collapsing a tubular neighbourhood of e to yield an equivariant stable map $\mathbf{S}^0 \rightarrow \mathbf{K}_J/\mathbf{N}_J(\mathbf{T})^\eta$, where $\mathbf{K}_J/\mathbf{N}_J(\mathbf{T})^\eta$ is the Thom-spectrum of the stable normal bundle. Including η into the trivial bundle, we get the K_J -equivariant transfer map given by the composite:

$$\mathbf{T}(J, e) : \mathbf{S}^0 \longrightarrow \mathbf{K}_J/\mathbf{N}_J(\mathbf{T})^\eta \longrightarrow \mathbf{K}_J/\mathbf{N}_J(\mathbf{T})^{\eta \oplus \tau} = \mathbf{K}_J/\mathbf{N}_J(\mathbf{T})_+,$$

where τ denotes the stable tangent bundle of $K_J/N_J(T)$. The collection of maps $\mathbf{T}(J, e)$ yield a map from the restriction of \mathbf{X} over the zero-skeleton of $|\mathcal{S}(A)|$ to the corresponding restriction of $\text{hocolim}_{J \in \mathcal{S}(A)} \mathbf{K}_J/\mathbf{N}_J(\mathbf{T})_+$. It remains to extend this map to the whole simplicial complex $|\mathcal{S}(A)|$. This is the point where the two properties stated above are crucial.

By property (2), we may identify the simplicial complex $|\mathcal{S}(A)|$ with a piecewise-affine subspace of \mathfrak{h} , with the property that the vertex corresponding to J_i is identified with the element ρ_{J_i} . Let $B|\mathcal{S}(A)|$ denote the barycentric subdivision of $|\mathcal{S}(A)|$. Working inductively with the faces, we may define a simplicial map: $\pi : B|\mathcal{S}(A)| \rightarrow |\mathcal{S}(A)|$, with the property $\pi(b(\Delta)) = J_k$, where $b(\Delta)$ denotes the vertex given by the barycenter of a k -dimensional face Δ of $|\mathcal{S}(A)|$ corresponding to a sequence of inclusions: $J_0 < \dots < J_k$. Similarly, define λ to be the map $\lambda : B|\mathcal{S}(A)| \rightarrow |\mathcal{S}(A)|$ with the property $\lambda(b(\Delta)) = J_0$.

Now consider the following face-preserving map \mathbf{T} over the k -simplex Δ :

$$\mathbf{T} : \Delta \times (\text{Emb}(K_{J_k}/N_{J_k}(T))_+ \wedge \mathbf{S}^0) \longrightarrow \text{hocolim}_{J \in \mathcal{S}(A)} \mathbf{K}_J/\mathbf{N}_J(\mathbf{T})_+,$$

$$\mathbf{T}(x, e) = (\pi(x), \iota(\lambda(x)) \circ \mathbf{T}(J_k, e)).$$

Here, given $\mu \in \Delta$, the map $\iota(\mu) \circ \mathbf{T}(J_k, e)$ denotes the composite:

$$\mathbf{S}^0 \longrightarrow \mathbf{K}_{J_k}/\mathbf{N}_{J_k}(\mathbf{T})^\eta \xrightarrow{\iota(\mu)} \mathbf{K}_{J_k}/\mathbf{N}_{J_k}(\mathbf{T})^{\eta \oplus \tau} = \mathbf{K}_{J_k}/\mathbf{N}_{J_k}(\mathbf{T})_+,$$

where $\iota(\mu)$ is the section of the bundle $\eta \oplus \tau \rightarrow \eta$ generated by the vector field given by the infinitesimal action of $\mu \in \mathfrak{h}$.

Notice that if J_i represents a vertex of $|\mathcal{S}(A)|$, then property (1) above, implies that the vectorfield on $K_{J_k}/N_{J_k}(T)$, generated by ρ_{J_i} vanishes exactly on $K_{J_i}/N_{J_i}(T)$, and hence the map $\iota(\rho_{J_i})$ collapses the complement of a tubular neighborhood of $K_{J_i}/N_{J_i}(T)$ to the basepoint, thereby showing that the above definition of \mathbf{T} extends $\mathbf{T}(J_i, e)$. Moreover, the above definition of \mathbf{T} is compatible with overlaps of faces, hence \mathbf{T} yields a face-preserving map of equivariant spectra from \mathbf{X} to $\text{hocolim}_{J \in \mathcal{S}(A)} \mathbf{K}_J/\mathbf{N}_J(\mathbf{T})_+$ over $|\mathcal{S}(A)|$. The upshot of this argument is that we have a face-preserving transfer map of equivariant spectra over $|\mathcal{S}(A)|$ given by a zig-zag:

$$\mathbf{T} : \text{hocolim}_{J \in \mathcal{S}(A)} \mathbf{S}^0 \xleftarrow{\sim} \mathbf{X} \longrightarrow \text{hocolim}_{J \in \mathcal{S}(A)} \mathbf{K}_J/\mathbf{N}_J(\mathbf{T})_+.$$

It is now easy to see that the induced transfer map:

$$\overline{\mathbf{T}} : \text{hocolim}_J \mathbf{E}\mathbf{K}_+ \wedge_{\mathbf{K}_J} \mathbf{S}^0 \longrightarrow \text{hocolim}_J \mathbf{E}\mathbf{K}_+ \wedge_{\mathbf{K}_J} (\mathbf{K}_J/\mathbf{N}_J(\mathbf{T})_+).$$

is indeed a stable retraction. This can be established by observing that $\overline{\mathbf{T}}$ induces a map of the respective Bousfield-Kan spectral sequences that compute the stable homotopy of the spectra $\mathbf{B}\mathbf{K}_+$ and $\mathbf{B}\mathbf{N}(\mathbf{T})_+$ respectively. The properties of $\overline{\mathbf{T}}$ ensure that this map is a retraction on the E_2 -term, and hence is a retraction. As a consequence, we have:

Theorem 11.1. *The above map $\overline{\mathbf{T}}$ is a stable retraction of $\mathbf{B}\mathbf{K}_+$ from $\mathbf{B}\mathbf{N}(\mathbf{T})_+$. In addition, $\overline{\mathbf{T}}$ is compatible with the stable retractions of each spectra $\mathbf{B}\mathbf{K}_{J_+}$ from $\mathbf{B}\mathbf{N}_J(\mathbf{T})_+$.*

Remark 11.2. *Let $Z(K) \subseteq T$ denote the center of K . Then the above construction shows that $\overline{\mathbf{T}}$ is equivariant with respect to the action of $BZ(K)$ on $\mathbf{B}\mathbf{K}_+$ and $\mathbf{B}\mathbf{N}(\mathbf{T})_+$. In particular, $\overline{\mathbf{T}}$ descends to stable transfers for central quotients of K .*

12. APPENDIX

In this section we establish some basic facts about the topology of the Kac-Moody groups K , and their classifying spaces BK .

Recall that a subset $J \subseteq I$ is called spherical if the subgroup $K_J \subseteq K$ is a compact Lie group. The poset of spherical subsets of I is denoted by $\mathcal{S}(A)$. In [K2] (Theorem A) it is shown that as an abstract group, K is an amalgamated product of subgroups of the form K_J , where $J \in \mathcal{S}(A)$ has cardinality at most two. In other words, the following canonical map is an isomorphism:

$$\operatorname{colim}_{J \in \mathcal{S}(A), |J| \leq 2} K_J \longrightarrow K,$$

where the colimit is taken in the category of groups. Now given $J \in \mathcal{S}(A)$ it is easy to see that K_J is generated by the groups K_j for $j \in J$. Hence, the map above factors through a sequence of two surjective maps:

$$\operatorname{colim}_{J \in \mathcal{S}(A), |J| \leq 2} K_J \longrightarrow \operatorname{colim}_{J \in \mathcal{S}(A)} K_J \xrightarrow{\varphi} K.$$

As a consequence we see that the map φ above is an isomorphism of (abstract) groups.

We now come to the question of topology on K . We refer the reader to [K1, Ku] for details regarding the Bruhat decomposition of K described below.

Let $N(T) \subset K$ denote the normalizer of T . Given $w \in W$, let $\tilde{w} \in N(T)$ denote any lift of w in $N(T)$. We will denote the space $B\tilde{w}B \cap K$ by Z_w . This is a well defined subspace of K homeomorphic, as a right T -space, to $\mathbb{C}^{l(w)} \times T$. Now for a generating reflection r_i , let $Y_i \subset Z_{r_i}$ be the subspace $\mathbb{C} \times \{1\} \subset \mathbb{C} \times T$ under the above identification. Then the group product in K induces a homeomorphism:

$$Z_w = Y_{i_1} \times \dots \times Y_{i_s} \times T,$$

where $w = r_{i_1} \dots r_{i_s}$ is a reduced expression. We also have the closure relation:

$$\overline{Z}_w = \prod_{v \leq w} Z_v.$$

With this structure, K becomes a T -CW complex, constructed by successively attaching T -cells. The topology is generated by the closed subspaces \overline{Z}_w . Hence a subspace $Z \subseteq K$ is closed if and only if $Z \cap \overline{Z}_w$ is closed for all $w \in W$. Now given $J \in \mathcal{S}(A)$, let $w_0 \in W_J$ denote the longest element. It follows from the closure relation that $\overline{Z}_{w_0} = K_J$ as compact subspaces of K .

Assume now that H is any topological group and that we are given a homomorphism $\phi : K \rightarrow H$ that restricts to a continuous map on each K_J for $J \in \mathcal{S}(A)$. Given an element $w \in W$, let $w = r_{i_1} \dots r_{i_s}$ be a reduced expression. Notice that ϕ extends to a canonical continuous map $\tilde{\phi}$ from the product $K_{i_1} \times \dots \times K_{i_s}$ to H given by the product of the individual restriction maps. Moreover, $\tilde{\phi}$ factors through the projection map from $K_{i_1} \times \dots \times K_{i_s}$ onto the subspace \overline{Z}_w . It follows that ϕ restricts to a continuous map on \overline{Z}_w . By the definition of the topology on K , we see and that ϕ is in fact a continuous homomorphism.

The upshot of the argument given above is that K is in fact the colimit of the groups K_J indexed over the poset $\mathcal{S}(A)$ in the category of *Topological Groups*. We conclude:

Theorem 12.1. *The topological group K has the following properties:*

- (a) K is a free T -CW complex of finite type under the right action of T . This structure is compatible with the CW structure on the homogeneous space K/T .
- (b) K is equivalent to the colimit, in the category of topological groups, of the compact Lie groups K_J indexed over the poset $\mathcal{S}(A)$.

Remark 12.2. *Since K is a T -CW complex, it is built by successively attaching T -cells. Decomposing T as a CW-complex, we see that K may be constructed by successively attaching (standard) cells. However, it fails to be a CW complex by virtue of the fact that the boundary of cells being attached may glue to cells of higher dimension. We will call a space built by attaching cells in a possibly non-sequential order a Cell Complex (there is some conflict in the literature on the terminology for such an object). Working inductively with the stages, it is easy to see that a cell complex is homotopy equivalent to a CW complex.*

We now consider the classifying space BK of K . Let us briefly recall its construction [M]. We let E_nK denote the n -fold join of K with itself given the quotient (weak) topology. Hence E_nK can be seen as a quotient of $\Delta^{n-1} \times K^{\times n}$. Using the fact that K is a cell complex of finite type, we see that E_nK is itself a cell complex of finite type, and that K acts continuously on E_nK . Let B_nK denote the quotient E_nK/K . It is easy to see that B_nK has the structure of a cell complex of finite type. Let BK be the cell complex given by the colimit of the spaces B_nK under the cellular inclusions $B_nK \subset B_{n+1}K$. Milnor shows [M](Section 5), that BK is a model for the classifying space of K . We conclude:

Theorem 12.3. *The classifying space BK of a Kac-Moody group has the homotopy type of a CW-complex. Consequently, the universal space EK is equivalent to a K -CW complex.*

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