

# THE MORAVA $K$ -THEORY OF SPACES RELATED TO $BO$

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ABSTRACT. We calculate the ( $p = 2$ ) Morava  $K$ -theory of all of the spaces in the connective Omega spectra for  $\mathbb{Z} \times BO$ ,  $BO$ ,  $BSO$ , and  $BSpin$ . This leads to a description of the ( $p = 2$ ) Brown-Peterson cohomology of many of these spaces. Of particular interest is the space  $BO\langle 8 \rangle$  and its relationship to  $BSpin$ .

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## 1. INTRODUCTION

We are interested in all of the spaces associated with  $BO$  and its connective covers. These are the spaces in the Omega spectra for  $\mathbb{Z} \times BO$ ,  $BO$ ,  $BSO$  and  $BSpin$ . There are various notations in the literature for these spaces and we have chosen to keep the notation,  $\underline{bo}_*$ , for the spectrum with  $\underline{bo}_0 = \mathbb{Z} \times BO$  (and  $\underline{bu}_*$  for the spectrum with  $\underline{bu}_0 = \mathbb{Z} \times BU$ ). However, in general we let  $\underline{BG}_*$  be the Omega spectrum with  $\underline{BG}_0 = BG$  rather than use, for example, the more traditional  $BO\langle 4 \rangle$  for  $BSpin$ . This notation works nicely with respect to the connective covers of various spaces. For example, the maps from left to right give all the connective covers:

$$(1.1) \quad \begin{array}{ccccccccc} \cdots & \longrightarrow & \underline{bo}_{i+8} & \longrightarrow & \underline{BSpin}_i & \longrightarrow & \underline{BSO}_i & \longrightarrow & \underline{BO}_i & \longrightarrow & \underline{bo}_i \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ & & K(\mathbb{Z}, i+8) & & K(\mathbb{Z}, i+4) & & K(\mathbb{F}_2, i+2) & & K(\mathbb{F}_2, i+1) & & K(\mathbb{Z}, i). \end{array}$$

For a space on the top line, the space to the left is the fibre induced by the vertical map to an Eilenberg-Mac Lane space below. We will compute  $K_*\underline{bo}_*$  and  $K_*\underline{BG}_*$  for  $K = K(n)$  ( $p = 2$ ) and  $BG = BO, BSO, BSpin$  and all the relevant maps. Recall that  $K(n)_*(-)$  is a generalized homology theory with coefficient ring  $K(n)_* \simeq \mathbb{F}_p[u_n, u_n^{-1}]$ , degree  $u_n = 2(p^n - 1)$ , known as the  $n$ -th Morava  $K$ -theory. Our calculations are in the spirit and tradition of Stong's calculations of the standard homology of the connective covers of  $BO$ , [Sto63]. On close examination of his paper and ours, we got the easy end of things.

The model for our results is the similar calculation of  $K_*\underline{bu}_*$  for all primes.

**Theorem 1.2** ([RWY98, Section 2.6]). *Let  $K = K(n)$  (for any prime  $p$ ). Let  $B = \underline{bu}_i$ ,  $i \geq 4$ . Let  $E \rightarrow B$  be a connective cover with fibre  $F$ . Then there is a short exact sequence of Hopf algebras:*

$$K_* \rightarrow K_*F \rightarrow K_*E \rightarrow K_*B \rightarrow K_*.$$

This is really only done for  $B = BSU = \underline{bu}_4$  in [RWY98] but it is not difficult to get the general result from this. Since  $K_*BSU$  is polynomial and  $K_*\underline{bu}_5$  is exterior this is split as algebras and since  $F$  is a finite Postnikov system with known homotopy its Morava  $K$ -theory is known from [HRW98] and [RW80].

Our first result looks very similar to this.

**Theorem 1.3.** *Let  $K = K(n)$  ( $p = 2$ ). Let  $B = \underline{bo}_i$ ,  $i \geq 4$ , or  $\underline{BG}_i$ ,  $i \geq 2$ ,  $G = O, SO$ , or  $Spin$ . Let  $E \rightarrow B$  be a connective cover with fibre  $F$ . Then there is a short exact sequence of Hopf algebras:*

$$K_* \rightarrow K_*F \rightarrow K_*E \rightarrow K_*B \rightarrow K_*.$$

Again, by Bott periodicity we know the homotopy of the fibre,  $F$ , and so by [HRW98] and [RW80] we know  $K_*F$ . Because of the massive redundancy in this theorem (e.g.  $\underline{bo}_{10} \rightarrow \underline{BO}_2$  is a covering), we need only compute the Morava  $K$ -theory of  $\underline{bo}_i$ ,  $i = 4, 5, 6, 7, 8$  and  $9$  and  $\underline{BO}_i$  for  $i = 2$  and  $3$ . All of the other spaces in the theorem are connective covers of these eight spaces. Corresponding theorems are true for odd primes also because these spaces are all pieces of spaces in  $\underline{bu}_*$  at an odd prime.

In the range of this theorem, all of the maps from the left in Diagram (1.1) are surjective in Morava  $K$ -theory and the vertical maps to Eilenberg-Mac Lane spaces are trivial. However, if  $i$  is low enough the Eilenberg-Mac Lane space must split off and the vertical map should be a surjection. Thus we have some “transition” spaces we must deal with.  $BU$  gives us a good example of this phenomenon.

Here, the  $\underline{bu}_i$  for  $i = 0, 1, 2, 3$ , and  $4$  are  $\mathbb{Z} \times BU$ ,  $U$ ,  $BU$ ,  $SU$ , and  $BSU$  respectively. Of course, if  $i$  is negative and even,  $\underline{bu}_i = \mathbb{Z} \times BU$ , and if negative and odd,  $\underline{bu}_i = U$ .  $K_*BU$  is polynomial and  $K_*U$  is exterior. The fibration  $\underline{bu}_2 \rightarrow \underline{bu}_0 \rightarrow \mathbb{Z}$  gives a (split) short exact sequence of Hopf algebras in Morava  $K$ -theory.  $K_*\underline{bu}_3$  is exterior and  $\underline{bu}_3 \rightarrow \underline{bu}_1 \rightarrow S^1$  also gives a (split) short exact sequence of Hopf algebras in Morava  $K$ -theory.  $K_*\underline{bu}_4$  is polynomial and  $\underline{bu}_4 \rightarrow \underline{bu}_2 \rightarrow K(\mathbb{Z}, 2)$  gives a short exact sequence in Morava  $K$ -theory which is not split, even as algebras.  $K_*\underline{bu}_5$  is exterior and  $K_*\underline{bu}_5 \rightarrow K_*\underline{bu}_3$  is surjective and the kernel is generated by  $n - 1$  exterior generators. These low spaces and maps constitute our transition spaces from (split) surjection to trivial. After this, the fibration  $K(\mathbb{Z}, i - 1) \rightarrow \underline{bu}_{i+2} \rightarrow \underline{bu}_i$ ,  $i \geq 4$ , gives a short exact sequence.

In the next section we will state all of our results for these transitions for  $BO$  precisely. It is a bit more complicated than the  $BU$  case. A variety of things happen. It is instructive to give an example here. Note that  $\underline{bo}_8$  is the space frequently referred to as  $BO\langle 8 \rangle$ . Recall from [RW80] that the fibration

$$(1.4) \quad K(\mathbb{F}_2, i) \longrightarrow K(\mathbb{Z}, i + 1) \xrightarrow{2} K(\mathbb{Z}, i + 1)$$

gives rise to a short exact sequence of Hopf algebras in Morava  $K$ -theory. The study of the Morava  $K$ -theory of the fibrations

$$K(\mathbb{Z}, 3) \longrightarrow \underline{bo}_8 (= BO\langle 8 \rangle) \longrightarrow \underline{BSpin}_0 (= BSpin) \longrightarrow K(\mathbb{Z}, 4)$$

gives, perhaps, the most interesting example of our transition spaces:

**Theorem 1.5.** *Let  $K = K(n)$  ( $p = 2$ ). The maps*

$$K(\mathbb{F}_2, 2) \rightarrow K(\mathbb{Z}, 3) \rightarrow BO\langle 8 \rangle \rightarrow BSpin \rightarrow K(\mathbb{Z}, 4) \xrightarrow{2} K(\mathbb{Z}, 4)$$

*give rise to an exact sequence of Hopf algebras in Morava  $K$ -theory:*

$$K_* \rightarrow K_*K(\mathbb{F}_2, 2) \rightarrow K_*K(\mathbb{Z}, 3) \rightarrow K_*BO\langle 8 \rangle \longrightarrow$$

$$K_*BSpin \rightarrow K_*K(\mathbb{Z}, 4) \xrightarrow{2^*} K_*K(\mathbb{Z}, 4) \rightarrow K_*.$$

where  $K_*BSpin$  is polynomial. As algebras,  $K_*BO\langle 8 \rangle$  is polynomial tensor with  $K_*K(\mathbb{Z}, 3)$ . Algebraically, we have the four term exact sequence of Hopf algebras:

$$K_* \rightarrow K_*K(\mathbb{Z}, 3) \rightarrow K_*BO\langle 8 \rangle \rightarrow K_*BSpin \rightarrow K_*K(\mathbb{F}_2, 3) \rightarrow K_*.$$

Note that when  $K = K(1)$  in the six term sequence, the two terms on the left and the two terms on the right are all trivial and we are reduced to a (well known) two term isomorphism. When  $K = K(2)$  the two terms on the right are still trivial but the two on the left show up. Since this work was done we have learned that Matthew Ando, Michael Hopkins and Neil Strickland had calculated  $K(2)^*(BO\langle 8 \rangle)$  in a rather different form: as ‘‘Sigma’’ structures on the trivial line bundle over the formal group. They never published this work which was originally part of [AHS01]. When  $K = K(n)$  with  $n > 2$  we get a real six term exact sequence. It was our attempt to understand the map  $BO\langle 8 \rangle \rightarrow BSpin$  for  $K(2)$  which got this project started. This result was one of the very last things to be understood. In a follow-up paper, [KLW], we show that  $K_*K(\mathbb{Z}, 3)$  splits off of  $K_*BO\langle 8 \rangle$  as Hopf algebras (as a special case of a general splitting theorem).

Our approach is a bit wishy-washy as we determine things are polynomial or exterior without getting our hands on generators. We prefer to call the approach *coordinate free*. It turns out we can pretty much do everything this way and it is actually quite a bit easier since we don’t have to keep track of all those little elements. However, in some cases, there is some use to having elements named and so we do that too. In general, the information is there to produce generators if the need arises. The above description of  $K_*BO\langle 8 \rangle$  and  $K_*BSpin$  are examples of the coordinate free approach.

Many people have a passion for  $BO\langle 8 \rangle$  and its relationship to  $BSpin$ . Much more detail can be extracted from our results for those who need it. It was known from [RWY98, Section 2.5] that  $K_*BO$  injects into  $K_*BU$  using the complexification map.  $K_*BU$  is  $K_*[b_1, b_2, \dots]$  as usual for any complex orientable theory.  $K_*K(\mathbb{Z}, 3)$  sits in  $K_*BO\langle 8 \rangle$  as a sub-Hopf algebra. The following Hopf algebras are all polynomial and the maps are all injections:

$$(1.6) \quad K_*BO\langle 8 \rangle // K_*K(\mathbb{Z}, 3) \rightarrow K_*BSpin \rightarrow K_*BSO \rightarrow K_*BO \rightarrow K_*BU.$$

Using the coordinate free approach, a particular finite Hopf algebra arose which got carried along through many of our spaces. However, going back and getting a detailed description of it, we discovered, very much to our surprise:

**Theorem 1.7.** *Let  $K = K(n)$  ( $p = 2$ ). The Hopf algebra cokernel,  $CK_1$ , of the Morava  $K$ -theory of the forgetful map  $BU \rightarrow BO$  is just*

$$K_*[b_1, b_2, \dots, b_{2^n-1}],$$

modulo the relations

$$0 = b_i^2 + \sum_{k \geq 0} u_n b_{2k} b_{2i-2k+1-2^n}$$

where  $b_0$  is the Hopf algebra unit and the coproduct is  $\psi(b_q) = \sum b_i \otimes b_{q-i}$ . This, in turn, is isomorphic to

$$K_*\left(\prod_{i>0} K(\mathbb{F}_2, i)\right).$$

This Hopf algebra is very well understood from [RW80] where the space is viewed as a graded space and the Hopf algebra is given the extra structure of a Hopf ring. There are no topological maps from  $BO$  to justify such a description; this is purely algebraic. In [Rao90], Rao computed  $K_*SO$  (among many other things) and discovered this Hopf algebra sitting there. He was able to evaluate the coalgebra structure but not the algebra structure and so was unable to make the identification above. However, his paper was a major influence on us in our work.

In [RWY98, Section 2.5],  $K_*BO$  was calculated using the Atiyah-Hirzebruch spectral sequence. In a more difficult calculation, it was shown there that it is the Hopf algebra kernel of the map  $(1-c)_*$  on  $K_*BU$  to itself where  $c$  is complex conjugation. Obviously the composition of this map with  $(1+c)_*$  is trivial. The Hopf algebras we are working with form an abelian category so we can talk about chain complexes and homology. We have another formulation of the above theorem.

**Theorem 1.8.** *Let  $K = K(n)$  ( $p = 2$ ). The homology of the middle term of the chain complex*

$$K_*BU \xrightarrow{(1+c)_*} K_*BU \xrightarrow{(1-c)_*} K_*BU$$

is

$$CK_1 = K_*\left(\prod_{i>0} K(\mathbb{F}_2, i)\right).$$

Techniques have been developed to take information about the Morava  $K$ -theory of a space and use it to compute the Brown-Peterson cohomology of that space. When this is possible the Brown-Peterson cohomology has some special properties. First, it is Landweber flat.  $M$  is Landweber flat if it has no  $p$ -torsion,  $M/(p)M$  has no  $v_1$  torsion,  $M/(p, v_1)M$  has no  $v_2$  torsion, etc., where  $BP^* \simeq \mathbb{Z}_{(p)}[v_1, v_2, \dots]$ . This property gives these spaces a completed Künneth isomorphism, [RWY98, Theorem 1.11, page 149], and makes them into *completed Hopf algebras*, [KW01, Section 6]. These computational techniques have been developed and refined in [RWY98], [Kas98], [Wil99], [Kas01], and [KW01]. In our case we must usually move to the 2-adically completed version of  $BP$  cohomology,  $BP_2^*(-)$ . As an example of our applications:

**Theorem 1.9.** *For the spaces and conditions of Theorem 1.3, with  $i$  even, we have a short exact sequence of completed Hopf algebras*

$$BP_2^* \leftarrow BP_2^*F \leftarrow BP_2^*E \leftarrow BP_2^*B \leftarrow BP_2^*.$$

and all are concentrated in even degrees.

The category of completed Hopf algebras is not an abelian category so the short exactness referred to is not automatically defined. However, we can talk about kernels and cokernels, so what we mean by short exact is that one map is the kernel of the other and the second map is the cokernel of the first. A similar theorem is proven for  $\underline{bu}_*$  in [RWY98].

Diagram (1.6) gives us surjections:

$$(1.10) \quad BP^*BSpin \leftarrow BP^*BSO \leftarrow BP^*BO \leftarrow BP^*BU.$$

$BP^*BU$  is  $BP^*[[c_1, c_2, \dots]]$ , where the  $c_i$  are the Conner-Floyd Chern classes.

**Theorem 1.11.**

(i) ([Wil84], see also [RWY98])

$$BP^*BO \simeq BP^*BU/(c_i - c_i^*)$$

where  $c_i^*$  is the complex conjugate of the Conner-Floyd Chern class.

(ii)

$$BP^*BSO \simeq BP^*BO/(c_1(\det))$$

where  $c_1(\det)$  is the first Chern class of the determinant bundle.

(iii) We have an exact sequence of completed Hopf algebras:

$$BP^* \leftarrow BP^*BSpin \leftarrow BP^*BSO \leftarrow BP^*K(\mathbb{F}_2, 2) \leftarrow BP^*.$$

Although the Conner-Floyd Chern classes generate  $BP^*BSpin$  we have been unable to determine a nice description.  $BP^*K(\mathbb{F}_2, 2)$  is completely described in [RWY98] and is known to have generators in degrees  $2(1 + 2^i)$ ,  $i > 0$ .

The fact that we are not in an abelian category complicates the study of the map  $BO\langle 8 \rangle \rightarrow BSpin$ . However, we do have theorems analogous to Theorem 1.5. We have to go to the 2-adic completion of  $BP$ . We remind the reader of the short exact sequence of completed (Landweber Flat) Hopf algebras from Diagram 1.4, [RWY98], and [KW01].

$$(1.12) \quad BP_2^* \leftarrow BP_2^*K(\mathbb{F}_2, i) \leftarrow BP_2^*K(\mathbb{Z}, i+1) \xleftarrow{2^*} BP_2^*K(\mathbb{Z}, i+1) \leftarrow BP_2^*$$

**Theorem 1.13.** *We consider the fibration sequence*

$$K(\mathbb{Z}, 3) \xrightarrow{i} BO\langle 8 \rangle \xrightarrow{p} BSpin \xrightarrow{k} K(\mathbb{Z}, 4).$$

$BP_2^*(-)$  for all of these spaces is Landweber Flat.

(i) *The cokernel of the map*

$$BP_2^*BSpin \rightarrow BP_2^*BO\langle 8 \rangle \quad \text{is} \quad BP_2^*K(\mathbb{Z}, 3).$$

(ii) *The induced map  $i^*$  is the composition of the surjection above with the injection  $2^*$ .*

(iii) *The cokernel of  $i^*$  is  $BP_2^*K(\mathbb{F}_2, 2)$ .*

(iv) *The kernel of the map*

$$BP_2^*BSpin \rightarrow BP_2^*BO\langle 8 \rangle \quad \text{is} \quad BP_2^*K(\mathbb{F}_2, 3).$$

(v) *The map  $k^*$  factors through the surjection  $BP_2^*K(\mathbb{Z}, 4) \rightarrow BP_2^*K(\mathbb{F}_2, 3)$ .*

(vi) *The kernel of  $k^*$  is isomorphic to  $BP_2^*K(\mathbb{Z}, 4)$ . It is the image of the injection  $2^*$  on  $BP_2^*K(\mathbb{Z}, 4)$ .*

Combining the results above we get legitimate exact sequences:

$$BP_2^* \leftarrow BP_2^*K(\mathbb{F}_2, 2) \leftarrow BP_2^*K(\mathbb{Z}, 3) \leftarrow BP_2^*BO\langle 8 \rangle$$

and

$$BP_2^*BSpin \leftarrow BP_2^*K(\mathbb{Z}, 4) \xleftarrow{2^*} K(\mathbb{Z}, 4) \leftarrow BP_2^*.$$

Again, we cannot describe the image of the injection  $BP_2^*K(\mathbb{F}_2, 3) \rightarrow BP_2^*BSpin$  in terms of Conner-Floyd Chern classes, but from [RWY98] we know that  $BP_2^*K(\mathbb{F}_2, 3)$  has generators in degrees  $2(1 + 2^i + 2^j)$ ,  $0 < i < j$ . We cannot state this theorem as a long exact sequence of completed Hopf algebras (yet). The issues are delicate and have to do with the topology on the  $BP$  cohomology. Thus

there is still some opportunity to improve upon our understanding of the map of  $BO\langle 8 \rangle \rightarrow BSpin$  in Brown-Peterson cohomology.

Whenever a finite Postnikov system shows up in our work we know its homotopy by Bott periodicity. In [HRW98] the Morava  $K$ -theory of such a finite Postnikov tower is shown to be, as Hopf algebras, just the Morava  $K$ -theory of the product of Eilenberg-Mac Lane spaces with the same homotopy. The Morava  $K$ -theory of these Eilenberg-Mac Lane spaces is computed explicitly in [RW80]. Thus, whenever the Morava  $K$ -theory of a finite Postnikov system shows up in our work then we can assume we know everything about it. Because the Morava  $K$ -theory of a finite Postnikov system is even degree, it is known, from [RWY98], to be Landweber Flat. However, it is not known if the Brown-Peterson cohomology also splits the same way Morava  $K$ -theory does or not. In the case of  $\underline{bu}_*$  the Brown-Peterson cohomology of these finite Postnikov systems is shown to split algebraically in [KW01]. The short exact sequence, and corresponding splitting are also shown for odd spaces. In this paper, we are unable to deal with the odd spaces in general and also cannot get an algebraic splitting of the sort known for  $\underline{bu}_*$  for our finite Postnikov systems.

In many cases we theoretically “know” everything about a Hopf algebra. That does not always translate into an explicit description of the Hopf algebra, or, equivalently, its Dieudonné module (see [SW98]).

To a large extent, this paper is a result of questions arising from [KL02] and [AHS01].

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The paper is organized as follows. In Section 2 we state the results of our calculations explicitly. Section 3 studies the map of  $K_*BU$  to  $K_*BO$  and proves Theorems 1.7 and 1.8. In Section 4 we review some facts about Hopf algebras and the bar spectral sequence. We settle down and prove most of our results in Section 5. Section 6 is dedicated to Brown-Peterson cohomology results.

## 2. STATEMENT OF RESULTS

**2.1. Introduction.** In this section we first give the Morava  $K$ -theory of all of our spaces and the description of all the maps of coverings. Then we give some of the relations with  $bu$ .

Many of our spaces are familiar spaces.

$$\begin{array}{ll}
 \underline{bo}_6 = Spin/SU & \underline{BSO}_1 = SU/Spin \\
 \underline{bo}_5 = SU/Sp & \underline{BSO}_0 = BSO \\
 \underline{bo}_4 = BSp & \underline{BSO}_{-1} = SO \\
 \underline{bo}_3 = Sp & \underline{BSO}_{-2} = O/U \\
 \underline{bo}_2 = Sp/U & \underline{BSpin}_0 = BSpin \\
 (2.1.1) \quad \underline{bo}_1 = U/O & \underline{BSpin}_{-1} = Spin \\
 \underline{bo}_0 = \mathbb{Z} \times BO & \underline{BSpin}_{-2} = SO/U \\
 \underline{BQ}_2 = Sp/SU & \underline{BSpin}_{-3} = U/Sp (= S^1 \times SU/Sp) \\
 \underline{BQ}_1 = SU/SO & \underline{BSpin}_{-4} = \mathbb{Z} \times BSp \\
 \underline{BQ}_0 = BO & \\
 \underline{BQ}_{-1} = O &
 \end{array}$$

*Remark 2.1.2.* Recall that for any Omega spectrum  $\{\underline{X}_*\}$  we have  $\Omega\underline{X}_{i+1} \simeq \underline{X}_i$ . When these spaces are looped down far enough they become redundant. For example,  $\underline{bo}_{-1} \simeq \underline{BO}_{-1}$ ,  $\underline{BO}_{-2} \simeq \underline{BSO}_{-2}$ ,  $\underline{BSO}_{-3} \simeq \underline{BSpin}_{-3}$ , and  $\underline{BSpin}_{-5} \simeq \underline{bo}_3$ . When  $i$  is small enough, i.e. negative enough, then  $\underline{X}_i = \underline{X}_{i+8}$ . All other negatively index spaces follow from these facts.

*Remark 2.1.3.* From Theorem 1.3 and Remark 2.1.2 we see that we need only compute the Morava  $K$ -theory of the following list of spaces:  $\underline{bo}_i$  for  $0 \leq i < 10$ ,  $\underline{BO}_i$  for  $-1 \leq i < 4$ ,  $\underline{BSO}_i$  for  $-2 \leq i < 2$ , and  $\underline{BSpin}_i$  for  $-4 \leq i < 2$ .

**2.2. Results for all spaces and covering maps.** Our first result to mention is the one stated in the introduction, Theorem 1.3. Next, we will describe our transition spaces and maps. Our starting point is the fairly easy fact, [RWY98, Section 2.5], that  $K_*BO$  is polynomial on even degree generators.

In our description, we need some special even degree finite Hopf algebras and short exact sequences which are split as Hopf algebras:

$$(2.2.1) \quad K_* \rightarrow CK_1 \rightarrow CK_0 \rightarrow K_*[\mathbb{F}_2] \rightarrow K_*$$

$$(2.2.2) \quad K_* \rightarrow CK_2 \rightarrow CK_1 \rightarrow K_*K(\mathbb{F}_2, 1) \rightarrow K_*$$

$$(2.2.3) \quad K_* \rightarrow CK_3 \rightarrow CK_2 \rightarrow K_*K(\mathbb{F}_2, 2) \rightarrow K_*$$

In our coordinate free approach these Hopf algebras came up without our knowing explicitly what they were. However, we will do them first as everything becomes easier when they are identified.

$$(2.2.4) \quad CK_j = K_*\left(\prod_{i \geq j} K(\mathbb{F}_2, i)\right).$$

**Definition 2.2.5.** Since we are working modulo 2 it is not automatic that exterior algebras have odd degree generators or polynomials have even degree generators. However, we want to set the convention, see Restriction A, Remark 4.5, that when we say exterior, we mean an exterior Hopf algebra on odd degree generators which we will generically denote by  $E$ . When we say polynomial, we mean a polynomial algebra on even degree generators which we will generically denote by  $P$ . Note the distinction between algebra and Hopf algebra. We tend to say little about the Hopf algebra structure but describe only the algebra structure. Frequently in our proofs, however, one can see the coalgebra structure. There will almost always be an infinite number of generators for  $E$  and  $P$  unless otherwise stated. When we say short exact we mean as Hopf algebras in Morava  $K$ -theory.

**Theorem 2.2.6** ( $\underline{bo}_i$  and  $\underline{BO}_i \rightarrow \underline{bo}_i$ ). *Let  $K = K(n)$  ( $p = 2$ ).*

- (i) *For  $i < 0$  we have  $\underline{bo}_i \simeq \underline{BO}_i$ .*
- (ii) *The fibration*

$$BO = \underline{BO}_0 \rightarrow \mathbb{Z} \times BO = \underline{bo}_0 \rightarrow \mathbb{Z}$$

*gives a split short exact sequence of Hopf algebras where  $K_*BO$  is polynomial.*

- (iii) *The fibration*

$$\underline{BO}_1 \rightarrow \underline{bo}_1 \rightarrow S^1$$

*gives a split short exact sequence of Hopf algebras where  $K_*\underline{BO}_1$  and  $K_*\underline{bo}_1$  are exterior.*

(iv) *The fibration*

$$\underline{BO}_2 \rightarrow \underline{bo}_2 \rightarrow K(\mathbb{Z}, 2)$$

*gives a short exact sequence where  $K_*\underline{BO}_2$  and  $K_*\underline{bo}_2$  are polynomial.*

(v) *The map*

$$K_*\underline{BO}_3 \rightarrow K_*\underline{bo}_3$$

*is a surjection, both are exterior and the kernel is an exterior Hopf algebra on  $n - 1$  generators.*

(vi) *For  $i \geq 4$  the fibration*

$$K(\mathbb{Z}, i - 1) \rightarrow \underline{BO}_i \rightarrow \underline{bo}_i$$

*gives a short exact sequence.*

(vii)  *$K_*\underline{bo}_4$  is polynomial.*

(viii)  *$K_*\underline{bo}_5$  is exterior.*

(ix)  *$K_*\underline{bo}_6$  is polynomial tensor  $CK_2$ .*

(x)  *$K_*\underline{bo}_7$  is exterior tensor  $CK_3$ .*

(xi)  *$K_*\underline{bo}_8$  is polynomial tensor  $K_*K(\mathbb{Z}, 3)$ .*

(xii)  *$K_*\underline{bo}_9$  is exterior tensor  $K_*K(\mathbb{Z}, 4)$ .*

**Theorem 2.2.7** ( $\underline{BO}_i$  and  $\underline{BSO}_i \rightarrow \underline{BO}_i$ ). *Let  $K = K(n)$  ( $p = 2$ ).*

(i) *For  $i < -1$  we have  $\underline{BO}_i \simeq \underline{BSO}_i$ .*

(ii) *The fibration*

$$SO = \underline{BSO}_{-1} \rightarrow O = \underline{BO}_{-1} \rightarrow \mathbb{F}_2$$

*gives a split short exact sequence of Hopf algebras*

$$K_* \rightarrow E \otimes CK_1 \rightarrow E \otimes CK_0 \rightarrow K_*[\mathbb{F}_2] \rightarrow K_*.$$

(iii) *The fibration*

$$BSO = \underline{BSO}_0 \rightarrow BO = \underline{BO}_0 \rightarrow K(\mathbb{F}_2, 1)$$

*gives a short exact sequence with  $K_*BSO$  and  $K_*BO$  both polynomial.*

(iv) *The map*

$$K_*\underline{BSO}_1 \rightarrow K_*\underline{BO}_1$$

*is a map of exterior algebras with both the kernel and the cokernel exterior on  $n - 1$  generators.*

(v) *For  $i \geq 2$  the fibration*

$$K(\mathbb{F}_2, i) \rightarrow \underline{BSO}_i \rightarrow \underline{BO}_i$$

*gives a short exact sequence.*

(vi)  *$K_*\underline{BO}_2$  is polynomial.*

(vii)  *$K_*\underline{BO}_3$  is exterior.*

**Theorem 2.2.8** ( $\underline{BSO}_i$  and  $\underline{BSpin}_i \rightarrow \underline{BSO}_i$ ). *Let  $K = K(n)$  ( $p = 2$ ).*

(i) *For  $i < -2$  we have  $\underline{BSO}_i \simeq \underline{BSpin}_i$ .*

(ii) *The fibration*

$$\underline{BSpin}_{-2} \rightarrow \underline{BSO}_{-2} \rightarrow \mathbb{F}_2$$

*gives a split short exact sequence of Hopf algebras*

$$K_* \rightarrow P \otimes CK_1 \rightarrow P \otimes CK_0 \rightarrow K_*[\mathbb{F}_2] \rightarrow K_*.$$

(iii) *The fibration*

$$Spin = \underline{BSpin}_{-1} \rightarrow SO = \underline{BSO}_{-1} \rightarrow K(\mathbb{F}_2, 1)$$

gives a split short exact sequence of Hopf algebras

$$K_* \rightarrow E \otimes CK_2 \rightarrow E \otimes CK_1 \rightarrow K_*K(\mathbb{F}_2, 1) \rightarrow K_*$$

(iv) *The fibration*

$$BSpin = \underline{BSpin}_0 \rightarrow BSO = \underline{BSO}_0 \rightarrow K(\mathbb{F}_2, 2)$$

gives a (non-split) short exact sequence with  $K_*BSpin$  and  $K_*BSO$  both polynomial.(v) *The map*

$$K_*\underline{BSpin}_1 \rightarrow K_*\underline{BSO}_1$$

is a map of exterior algebras with both the kernel and the cokernel exterior on  $\binom{n-1}{2}$  generators.(vi) *For  $i \geq 2$  the fibration*

$$K(\mathbb{F}_2, i+1) \rightarrow \underline{BSpin}_i \rightarrow \underline{BSO}_i$$

gives a short exact sequence.

The exact sequence

$$K_* \rightarrow K_*BSpin \rightarrow BSO \rightarrow K(\mathbb{F}_2, 2) \rightarrow K_*$$

occurs in [KL02] for  $K = K(2)$ .**Theorem 2.2.9** ( $\underline{BSpin}_i$  and  $\underline{bo}_{i+8} \rightarrow \underline{BSpin}_i$ ). *Let  $K = K(n)$  ( $p = 2$ ).*(i) *For  $i < -4$  we have  $\underline{BSpin}_i \simeq \underline{bo}_{i+8}$ .*(ii) *The fibration*

$$\underline{bo}_4 \rightarrow \underline{BSpin}_{-4} \rightarrow \mathbb{Z}$$

gives a split short exact sequence with  $K_*\underline{BSpin}_{-4} \simeq P \otimes K_*[\mathbb{Z}]$ .(iii) *The fibration*

$$\underline{bo}_5 \rightarrow \underline{BSpin}_{-3} \rightarrow S^1$$

gives a split short exact sequence of exterior Hopf algebras.

(iv) *The fibration*

$$\underline{bo}_6 \rightarrow \underline{BSpin}_{-2} \rightarrow K(\mathbb{Z}, 2)$$

gives a (non-split) short exact sequence with  $K_*\underline{BSpin}_{-2} \simeq P \otimes CK_1$ .We need an injection of short exact sequences to explain what happens with  $CK$ .

$$\begin{array}{ccccccc} K_* & \longrightarrow & CK_2 & \longrightarrow & CK_1 & \longrightarrow & K_*K(\mathbb{F}_2, 1) \longrightarrow K_* \\ & & \downarrow & & \downarrow & & \downarrow \\ K_* & \longrightarrow & K_*\underline{bo}_6 & \longrightarrow & K_*\underline{BSpin}_{-2} & \longrightarrow & K_*K(\mathbb{Z}, 2) \longrightarrow K_* \end{array}$$

(v) *The fibration*

$$\underline{bo}_7 \rightarrow Spin = \underline{BSpin}_{-1} \rightarrow K(\mathbb{Z}, 3)$$

is more complicated. There is an exterior Hopf algebra on  $n-1$  generators,  $E'$ . The maps

$$\underline{bo}_7 \twoheadrightarrow Spin = \underline{BSpin}_{-1} \twoheadrightarrow K(\mathbb{Z}, 3) \xrightarrow{2} K(\mathbb{Z}, 3)$$

give rise to an exact sequence of Hopf algebras:

$$K_* \longrightarrow E' \longrightarrow K_* \underline{bo}_7 \longrightarrow$$

$$K_* Spin \longrightarrow K_* K(\mathbb{Z}, 3) \xrightarrow{2*} K_* K(\mathbb{Z}, 3) \longrightarrow K_*.$$

and we have that  $K_* Spin$  is  $E \otimes CK_2$ . Another way to view this is

$$K_* \rightarrow E' \rightarrow E_1 \otimes CK_3 \rightarrow E \otimes CK_2 \rightarrow K_* K(\mathbb{F}_2, 2) \rightarrow K_*$$

(vi) The fibration maps associated with  $\underline{bo}_8 \rightarrow \underline{BSpin}_0$  are described in Theorem 1.5.  $K_* \underline{BSpin}_0 = K_* BSpin$  is polynomial.

(vii) The map

$$K_* \underline{bo}_9 \rightarrow K_* \underline{BSpin}_1$$

has kernel the  $K_* K(\mathbb{Z}, 4)$  in  $K_* \underline{bo}_9$  and  $\binom{n-1}{3}$  exterior generators.  $K_* \underline{BSpin}_1$  is exterior and all but  $\binom{n-1}{3}$  generators are in the image.

(viii) For  $i \geq 2$  the fibration

$$K(\mathbb{Z}, i+3) \rightarrow \underline{bo}_{i+8} \rightarrow \underline{BSpin}_i$$

gives a short exact sequence.

**2.3. Connections to  $BU$ .** There is a stable cofibration:

$$(2.3.1) \quad \Sigma bo \rightarrow bo \rightarrow bu$$

which we are unclear on what the most appropriate reference is. However, an unstable version seems to be due to Reg Wood, reproduced in [And64]. The stable connective version follows. It appears, for example, in [Dav72, page 186]. When you take various connective covers and look at the Omega spectra, this gives a number of fibrations involving our spaces and those for  $bu$ .

(2.3.2)

$$\begin{array}{ccccccccc} \underline{bo}_{k+1} & \rightarrow & \underline{bo}_k & \rightarrow & \underline{bu}_k & \rightarrow & \underline{bo}_{k+2} & \rightarrow & \underline{bo}_{k+1} \\ \underline{bo}_{k+1} & \rightarrow & \underline{BO}_k & \rightarrow & \underline{bu}_{k+2} & \rightarrow & \underline{bo}_{k+2} & \rightarrow & \underline{BO}_{k+1} \\ \underline{BO}_{k+1} & \rightarrow & \underline{BSO}_k & \rightarrow & \underline{bu}_{k+4} & \rightarrow & \underline{BO}_{k+2} & \rightarrow & \underline{BSO}_{k+1} \\ \underline{BSO}_{k+1} & \rightarrow & \underline{BSpin}_k & \rightarrow & \underline{bu}_{k+4} & \rightarrow & \underline{BSO}_{k+2} & \rightarrow & \underline{BSpin}_{k+1} \\ \underline{BSpin}_{k+1} & \rightarrow & \underline{bo}_{k+8} & \rightarrow & \underline{bu}_{k+6} & \rightarrow & \underline{BSpin}_{k+2} & \rightarrow & \underline{bo}_{k+9} \end{array}$$

There are also maps between these fibrations (up, the way they are written).

We have described the Morava  $K$ -theory of each of the spaces in all of these fibrations. To do justice we should also describe all of the maps involved. However, each space now comes equipped with several maps into and out of it and the burden of stating the results is too much for a reasonable length paper. Knowing the answers for all the spaces is usually enough to allow the reader to evaluate any maps the reader might need. To calculate the Morava  $K$ -theory of the various spaces we sometimes do need to resort to information in these fibrations and so we will state those results which come up naturally. In addition we also give the answers to some of the other more interesting cases.

Perhaps more important than what we need are a couple of results which were fundamental to the start of this whole project.

There are two short exact sequences, both of which are split exact as polynomial algebras, which we knew from before:

$$(2.3.3) \quad K_* \rightarrow K_*\underline{BO}_0 \rightarrow K_*\underline{bu}_2 \rightarrow K_*\underline{bo}_2 \rightarrow K_*,$$

which comes from  $BO \rightarrow BU \rightarrow Sp/U$ , and

$$(2.3.4) \quad K_* \rightarrow K_*\underline{bo}_2 \rightarrow K_*\underline{bu}_2 \rightarrow K_*\underline{bo}_4 \rightarrow K_*,$$

which comes from  $Sp/U \rightarrow BU \rightarrow BSp$ . The first is at the bottom of the page, [RWY98, Section 2.5, p. 161]. The second is in the middle of the page, [RWY98, Section 2.5, p. 162]. They both follow quickly from Theorem 4.8. The homologies of  $\underline{bo}_2$  and  $\underline{bo}_4$  are both polynomial in even degree so the same holds for  $K(n)_*(-)$  which gives the splitting.

**Theorem 2.3.5.** *Let  $K = K(n)$  ( $p = 2$ ).*

- (i) *In the fibration  $BU \rightarrow BO \rightarrow SO$ , the image of  $K_*BO \rightarrow K_*SO$  is the cokernel of  $K_*BU \rightarrow K_*BO$  which is  $CK_1$ .*
- (ii) *The map  $K_*U \rightarrow K_*SO$  hits all exterior generators.*
- (iii) *In the fibration  $SU \rightarrow Spin \rightarrow Spin/SU (= \underline{bo}_6)$ , the image of  $K_*Spin \rightarrow K_*\underline{bo}_6$  is the cokernel of  $K_*SU \rightarrow K_*Spin$  which is  $CK_2$ .*
- (iv) *In the fibration  $\underline{bo}_6 \rightarrow \underline{bu}_4 \rightarrow BSpin$ , the polynomial part of  $K_*\underline{bo}_6$  is split injective as algebras to the polynomial algebra  $K_*\underline{bu}_4$ .*
- (v) *In the fibration  $\underline{bu}_4 \rightarrow BSpin \rightarrow \underline{bo}_7$ , the cokernel of  $K_*\underline{bu}_4 \rightarrow K_*BSpin$  is  $CK_3$  which injects into  $K_*\underline{bo}_7$ .*
- (vi) *In the fibration  $\underline{bo}_6 \rightarrow \underline{bu}_6 \rightarrow \underline{bo}_8$ , the kernel of  $K_*\underline{bo}_6 \rightarrow K_*\underline{bu}_6$  is  $CK_3$ . The polynomial part of  $K_*\underline{bo}_6$  is split injective to the polynomial part of  $K_*\underline{bu}_6$  (which is  $K_*\underline{bu}_4$ ) and the  $K_*K(\mathbb{F}_2, 2)$  of  $CK_2$  injects to the  $K_*K(\mathbb{Z}, 3)$  part of  $K_*\underline{bu}_6$ .*
- (vii) *In the fibrations  $U \rightarrow SO \rightarrow \underline{BSpin}_{-2} \rightarrow \underline{bu}_2 (= BU)$  the image of  $K_*SO$  in  $K_*\underline{BSpin}_{-2}$  is the cokernel of  $K_*U \rightarrow K_*SO$ ,  $CK_1$ , and the cokernel of  $K_*SO \rightarrow K_*\underline{BSpin}_{-2}$  is polynomial which is split injective as algebras to the polynomial algebra  $K_*\underline{bu}_2$ .*

There are a number of associated short exact sequences which, if not useful, certainly have novelty value. We present them here even though we only need a few of them in our proofs. In fact, we will not prove most of them.

**Theorem 2.3.6.** *Let  $K = K(n)$  ( $p = 2$ ). The following fibrations give short exact sequences of Hopf algebras in  $K_*(-)$ :  $\underline{bo}_i \rightarrow \underline{bu}_i \rightarrow \underline{bo}_{i+2}$ ,  $\underline{BO}_i \rightarrow \underline{bu}_{i+2} \rightarrow \underline{bo}_{i+2}$ , and  $\underline{BSO}_i \rightarrow \underline{bu}_{i+4} \rightarrow \underline{BO}_{i+2}$  for  $i = 0, 1, 2, 3, 4$  and 8, and  $BSpin \rightarrow \underline{bu}_4 \rightarrow \underline{BSO}_2$  and  $\underline{bo}_8 \rightarrow \underline{bu}_6 \rightarrow \underline{BSpin}_2$ .*

### 3. $K_*BU$ AND $K_*BO$

We need a deeper understanding of  $K_*BO$  and its relationship to  $K_*BU$ .

We inherit a good deal of knowledge from [RWY98, Section 2.5] where  $K_*BO$  is shown to be a polynomial algebra using the Atiyah-Hirzebruch spectral sequence. It is easy to reproduce this calculation here.  $H_*(BO) \simeq \mathbb{F}_2[b_1^R, b_2^R, \dots]$  (modulo 2 of course). The first differential in the Atiyah-Hirzebruch spectral sequence for  $K_*BO$  ( $K = K(n)$ ) is obtained from the action of Milnor's  $Q_n$  in  $H_*(RP^\infty)$ . This takes  $b_{2^n+1+2k}^R$  to  $b_{2k+1}^R$  and so the spectral sequence collapses to the polynomial algebra

$$(3.1) \quad K_*[b_2^R, b_4^R, \dots, b_{2^{n+1}-2}^R] \otimes K_*[(b_{2^i}^R)^2 : i \geq 2^n].$$

The Atiyah-Hirzebruch spectral sequence for  $K_*BU$  collapses to the polynomial algebra  $K_*[b_1, b_2, \dots]$ . The complexification map,  $\mathbb{C}$ , on the Atiyah-Hirzebruch spectral sequence takes  $b_{2i}^R$  to  $b_i$  so we see, quite easily, that  $K_*BO \rightarrow K_*BU$  injects.

A much more difficult relation is also shown in [RWY98, Section 2.5]. Let  $c : BU \rightarrow BU$  be the complex conjugation. It is shown there that  $K_*BO$  is the Hopf algebra kernel of  $(1 - c)_*$ :

$$(3.2) \quad K_* \longrightarrow K_*BO \xrightarrow{\mathbb{C}_*} K_*BU \xrightarrow{(1-c)_*} K_*BU$$

This was used, in [RWY98], to reprove the result of [Wil84] that  $BP^*(BO) \simeq BP^*(BU)/(c_i - c_i^*)$  where the  $c_i$  are the Conner-Floyd Chern classes and the  $c_i^*$  are their conjugates.

We need more information. The following diagram gives us a start where  $\mathbb{R}$  is the forgetful map.

$$\begin{array}{ccccccc} K_*BU & \xrightarrow{\mathbb{R}_*} & K_*BO & \xrightarrow{\mathbb{C}_*} & K_*BU & \xrightarrow{(1-c)_*} & K_*BU \\ & & & & \nearrow & & \\ & & & & (1+c)_* & & \end{array}$$

Clearly the composition  $(1 - c)_*(1 + c)_*$  is trivial, so  $(1 + c)_*$  factors through the kernel of  $(1 - c)_*$ , which is  $K_*BO$ . Thus we can use  $(1 + c)_*$  to evaluate  $\mathbb{R}_*$  and calculate our Hopf algebra cokernel,  $CK_1$ . We need a better understanding of  $K_*BO$  as it sits inside of  $K_*BU$ .

We will need to make some of our calculations in connective Morava  $K$ -theory. Let  $k = k(n)$  with  $k(n)_* \simeq \mathbb{F}_2[u_n]$ . Since  $k(n)$  is a complex oriented theory, we still have  $k(n)_*BU \simeq k(n)_*[b_1, b_2, \dots]$ .

We need to know how  $c_*$  behaves on the generators of  $k(n)_*BU$ .

**Corollary 3.3.** *Let  $c : BU \rightarrow BU$  be complex conjugation. Then, for  $b_i \in k(n)_{2i}BU$ , ( $p = 2$ )*

$$\begin{aligned} c_*b_i &= b_i && \text{for } i < 2^n \\ c_*b_{2^n+i} &= b_{2^n+i} && \text{modulo } (u_n^2) \text{ for } i \text{ odd} \\ c_*b_{2^n+2q} &= b_{2^n+2q} + u_nb_{2q+1} && \text{modulo } (u_n^2). \end{aligned}$$

*Proof.* Because the diagram

$$\begin{array}{ccc} CP^\infty & \xrightarrow{c} & CP^\infty \\ \downarrow & & \downarrow \\ BU & \xrightarrow{c} & BU \end{array}$$

commutes, it is enough to prove this formula on the  $\beta_i \in k(n)_{2i}CP^\infty$  because they define the  $b_i \in k(n)_{2i}BU$ . This follows immediately from the next result.  $\square$

**Theorem 3.4.** *Let  $c : CP^\infty \rightarrow CP^\infty$  be complex conjugation. Then, for  $\beta_q \in k(n)_{2q}CP^\infty$ , ( $p = 2$ )*

$$c_*(\beta_q) = \sum_{i=0}^{2^n-1} u_n^i \binom{q - i(2^n - 1)}{i} \beta_{q-i(2^n-1)} \quad \text{modulo } u_n^{2^{n-1}+1}.$$

We do this for future reference even though we only need the first two terms. The proof is no harder to do this extra bit. The calculation could be carried out even further and we'll point out how.

**Theorem 3.5** (see [BP03]). *Let  $F$  be the formal group law for  $K = K(n)$  ( $p = 2$ ). For  $n > 1$*

$$x +_F y = x + y + u_n x^{2^{n-1}} y^{2^{n-1}} \quad \text{modulo } y^{2^{2(n-1)}}.$$

**Theorem 3.6** ([KL02]). *Let  $K = K(n)$  ( $p = 2$ ).*

$$-_F x = x +_F \sum_{j>0}^F u_n^{(2^{j^n}-1)/(2^n-1)} x^{2^{j^n}}.$$

*Proof.* The proof is by induction using the fact that  $x +_F x = u_n x^{2^n}$ . We add  $x$  (formally) to the above formula, it starts with  $x +_F x$ . The next step is  $u_n x^{2^n} +_F u_n x^{2^n} = u_n (u_n x^{2^n})^{2^n} = u_n^{1+u_n^{2^n}} x^{2^n}$ . Continue. Each step gives the next term and so the next step can also be evaluated.  $\square$

We now combine the last two results to get an explicit formula:

**Proposition 3.7.** *Let  $K = K(n)$  ( $p = 2$ ).*

$$-_F x = x + u_n x^{2^n} \quad \text{modulo } u_n^{1+2^{n-1}}.$$

*Proof.* Modulo  $u_n^{1+2^{n-1}}$ ,  $-_F x = x +_F u_n x^{2^n}$ . Using our formula for the formal group law, this is

$$x + u_n x^{2^n} + u_n x^{2^{n-1}} (u_n x^{2^n})^{2^{n-1}}$$

plus terms with  $u_n^{2^{2(n-1)}}$  in them. Modulo  $u_n^{1+2^{n-1}}$  this is just  $x + u_n x^{2^n}$ .  $n = 1$  is direct.  $\square$

*Remark 3.8.* A little more care and we can get:

$$-_F x = x + u_n x^{2^n} + u_n^{1+2^{n-1}} x^{2^{n-1}(1+2^n)} + u_n^{1+2^n} x^{2^{2^n}} \quad \text{modulo } u_n^{2^{2(n-1)}}.$$

*Proof of Theorem 3.4.* The complex conjugate on complex projective space is just the  $H$ -space inverse. If  $c_*(\beta_q) = \sum_i a_i u_n^i \beta_{q-i(2^n-1)}$  we can evaluate the  $a_i$  as follows:

$$\begin{aligned} a_i u_n^i &= \langle c_*(\beta_q), x^{q-i(2^n-1)} \rangle = \langle \beta_q, c^*(x)^{q-i(2^n-1)} \rangle = \\ &= \langle \beta_q, (x + u_n x^{2^n})^{q-i(2^n-1)} \rangle = \binom{q-i(2^n-1)}{i} u_n^i. \end{aligned}$$

$\square$

**Theorem 3.9.** *Let  $p = 2$ . Consider the map  $(1+c)_*$  on  $k(n)_*BU$ . Then (with  $b_0$  the Hopf algebra unit)*

$$(1+c)_*(b_{2q}) = b_q^2 + \sum_{k \geq 0} u_n b_{2k} b_{2q-2k+1-2^n} \quad \text{modulo } (u_n^2).$$

*Remark 3.10.* It will be important for us later on that these results are precise, not just modulo  $u_n^2$ , when  $2q < 2^{n+1}$ , this is because in  $k(n)$  theory there are no negative degree elements. Note also that the sum is trivial when  $2q < 2^n$ .

*Proof.* Let  $\psi$  be the  $k(n)_*BU$  Hopf algebra coproduct, and  $m$  the multiplication. The map  $(1+c)_*$  is defined by:

$$k(n)_*BU \xrightarrow{\psi} k(n)_*BU \otimes k(n)_*BU \xrightarrow{1_* \otimes c_*} k(n)_*BU \otimes k(n)_*BU \xrightarrow{m} k(n)_*BU.$$

We evaluate, modulo  $u_n^2$ ,

$$(1+c)_*(b_{2q}) = \sum_{i+j=2q} b_i c_*(b_j) = \sum_{i+j=2q} b_i b_j + \sum_{k \geq 0} u_n b_{2k} b_{2q-2k+1-2^n}.$$

The first sum is symmetric and so zero except for the middle term when we get  $b_q^2$ . This with the final sum is our desired result.  $\square$

**Theorem 3.11.** *Let  $p = 2$ . The kernel of*

$$(1-c)_* : k(n)_*BU \rightarrow k(n)_*BU$$

*contains the following elements:*

$$b_i \quad \text{for } i < 2^n$$

*and elements  $z_q \in k(n)_{2^{n+1}+2q}BU$  with*

$$z_q = b_{2^n+q}^2 + \sum_{k \geq 0} u_n b_{2k} b_{2^n+2q-2k+1} \quad \text{modulo } (u_n^2).$$

*Proof.* Let  $\chi$  be the  $k(n)_*BU$  Hopf algebra conjugation. We first change the problem to finding elements in the kernel of  $(c-1)_*$ . This is easier to compute with and it has the same kernel. The map  $(c-1)_*$  is just

$$k(n)_*BU \xrightarrow{\psi} k(n)_*BU \otimes k(n)_*BU \xrightarrow{c_* \otimes \chi} k(n)_*BU \otimes k(n)_*BU \xrightarrow{m} k(n)_*BU.$$

On  $b_i$ ,  $i < 2^n$ , this is

$$(c-1)_*(b_i) = \sum c_*(b_j) \chi(b_{i-j}) = \sum b_j \chi(b_{i-j}) = 0$$

because  $\chi$  satisfies this formula. So, we have the first elements verified in the kernel. The others are a little harder to come by. We use the fact that the composition  $(1-c)(1+c)$  is obviously trivial. If we produce our  $z_i$  in the image of  $(1+c)_*$  then we know that they are in the kernel of  $(1-c)_*$ . We have already done this in Theorem 3.9.  $\square$

Any element  $x \in K_*BU$  can be written as a sum of monomials in the  $b_i$  with coefficients in  $K(n)_* = \mathbb{F}_2[u_n, u_n^{-1}]$ . Infinite sums are not allowed in homology so by multiplying enough times by  $u_n$  we can replace  $x$  with an element written with only non-negative powers of  $u_n$ . In fact, every element of  $K_*X$  can be written, up to a unit, in terms of standard homology and sums with coefficients non-negative powers of  $u_n$ . We can always arrange that the first term has no  $u_n$  in it. In this sense, the Atiyah-Hirzebruch spectral sequence describes the ‘‘lead’’ term of an element. Our next result calculates the next non-trivial term.

**Theorem 3.12.** *Let  $K = K(n)$  ( $p = 2$ ).  $K_*BO$  sits inside  $K_*BU$  as the kernel of  $(1-c)_*$  as a sub-polynomial Hopf algebra. The generators can be written as  $b_i$  for  $0 < i < 2^n$  (recall that  $b_0$  is the Hopf algebra unit) and*

$$z_q = b_{2^n+q}^2 + \sum_{k \geq 0} u_n b_{2k} b_{2^n+2q-2k+1} + u_n^2 w_q \quad \text{for } q \geq 0$$

*where  $w_q$  is written with only non-negative powers of  $u_n$ .*

*Proof.* Inverting  $u_n$  to get  $K(n)_*BU$  from  $k(n)_*BU$  certainly preserves elements in the kernel. We have identified, in Theorem 3.11, these elements as being in the kernel in Diagram (3.2). Furthermore, we can see they are the generators of  $K_*BO$  from the Atiyah-Hirzebruch spectral sequence. See equation (3.1).  $\square$

Note the term  $u_nb_{2^n+2q+1}$ , which, since  $u_n$  is a unit, shows that  $z_q$  in  $K_*BO$  maps to a generator of  $K_*BU$  as well. This is not obvious from the Atiyah-Hirzebruch spectral sequence but is known from equation (2.3.3).

The image of the map  $(c-1)_*$  is the same as for  $(1-c)_*$  but it is a bit easier to compute with so we use it to compute:

**Proposition 3.13.** *Let  $p = 2$ . Modulo decomposables and  $u_n^2$ ,  $(c-1)_* : k(n)_*BU \rightarrow k(n)_*BU$  maps:*

$$(c-1)_*(b_{2^n+2q}) \rightarrow u_nb_{2q+1}.$$

*Proof.*

$$\begin{aligned} (c-1)_*(b_{2^n+2q}) &= \sum_{i+j=2^n+2q} c_*(b_i)\chi(b_j) \\ &= c_*(b_{2^n+2q}) + \chi(b_{2^n+2q}) \quad \text{modulo decomposables} \\ &= b_{2^n+2q} + u_nb_{2q+1} + b_{2^n+2q} = u_nb_{2q+1}. \end{aligned}$$

$\square$

**Theorem 3.14.** *Let  $K = K(n)$  ( $p = 2$ ). The sequence*

$$K_*BU \xrightarrow{(1-c)_*} K_*BU \xrightarrow{(1+c)_*} K_*BU$$

*is exact at the middle term.*

*Proof.* We know the composition is trivial. Because of Theorem 3.9 we know that  $(1+c)_*$  maps the  $b_{2q}$  to a sub-polynomial algebra and there are no relations. The biggest the kernel could be is the odd indexed generators (plus tails). But by Proposition 3.13 these are all in the image of  $(1-c)_*$ .  $\square$

*Proof of Theorem 1.7.* We know  $K_*BO$  from Theorem 3.12. We also know that  $(1+c)_*$  gives us the map of  $K_*BU$  to  $K_*BO$  sitting inside of  $K_*BU$ . From Theorem 3.9 we know that all of the elements  $z_q$  are in the image. So, our cokernel is generated by the  $b_i$ ,  $i < 2^n$ . Theorem 3.9 also gives us the relations for the squares of these elements. However, we have to be sure there are no more relations. The map  $(1+c)_*$  has only been evaluated on the  $b_{2q}$  at this stage so we must worry about the odd indexed  $b_{2q+1}$ . However, we have just shown that they are in the image of  $(1-c)_*$  (Proposition 3.13) and so they are killed by  $(1+c)_*$ . More precisely, we showed that an element with  $b_{2q+1}$  as lead term is in the kernel of  $(1+c)_*$ . This implies that  $(1+c)_*b_{2q+1}$  is in the image of the sub-algebra generated by the even  $b_{2k}$ . Thus there are no more relations.

The proof that this cokernel Hopf algebra is isomorphic to the Morava  $K$ -theory of a product of Eilenberg-Mac Lane spaces requires an intimate knowledge of the later from [RW80]. We review that now. We begin by introducing some bad notation just for the purpose of this proof. We let  $K_i = K(\mathbb{F}_2, i)$  and our goal is to describe  $K_*(\prod_i K_i)$ . We have a map  $K_1 \rightarrow CP^\infty$  making  $K_*K_1$  a sub-Hopf algebra of  $K_*CP^\infty$  on elements  $a_i \rightarrow b_i$  for  $i < 2^n$ . This gives our usual coproduct. The squares of these elements are  $a_i^2 = 0$  except when  $i = 2^{n-1}$  when we get  $u_n a_1$ .

We designate special elements with a change of notation:  $a_{(j)} = a_{2^j}$ . We use the maps  $K_i \wedge K_j \rightarrow K_{i+j}$  to define elements

$$a_I = a_{(i_1)} \circ a_{(i_2)} \circ \cdots \circ a_{(i_k)} \in K_* K_k$$

where  $0 \leq i_1 < i_2 < \cdots < i_k < n$ . It is important to note the degree of this is  $2 \sum 2^{i_j}$  and that there is one for every  $0 < 2i < 2^{n+1}$ . Thus the size of the exterior algebra on this is precisely the previously observed size of the cokernel  $CK_1$ . There are rules for squaring these elements. If  $i_k \neq n-1$  then  $a_I^2 = 0$ . If  $i_k = n-1$  then

$$a_I^2 = u_n a_{(0)} \circ a_{(i_1+1)} \circ a_{(i_2+1)} \circ \cdots \circ a_{(i_{k-1}+1)}.$$

Thus we have that  $CK_1$  really is the same size as  $K_*(\prod_i K_i)$ . We have described this in terms of the  $a_I$  and the above relations. We need also to point out that the primitives are given by the  $a_I$  with  $i_1 = 0$  so there is one for every  $0 < 2i < 2^{n+1}$  with  $i$  odd.

We will construct a Hopf algebra isomorphism from  $CK_1$  to  $K_*(\prod_i K_i)$ . We begin this process with a map of coalgebras from  $K_*\{b_1, b_2, \dots, b_{2^{n-1}}\}$  to  $K_*(\prod_i K_i)$ . The coproduct is given by  $\psi(b_j) = \sum b_{j-i} \otimes b_i$ . Likewise for the  $a_j$ . We could map  $b_j$  to  $a_j$ . To get our isomorphism though, we have to modify this map a little with the primitives in the image. Our map is constructed inductively on  $b_i$ . Clearly we send  $b_1$  to  $a_{(0)}$ . If  $i$  is even then the coproduct determines where our  $b_i$  must go and there is somewhere to go since  $K_*(\prod_i K_i)$  is cofree as a coalgebra in this range. If  $i$  is odd then we have options. The coproduct can determine where  $b_i$  must go, but only up to the primitives. We can choose to include the primitive or not, and we choose to include it as a term in our image. We can now use the algebra structure to extend our coalgebra map to a surjective Hopf algebra map

$$K_*[b_1, b_2, \dots, b_{2^{n-1}}] \rightarrow K_*\left(\prod_{i>0} K_i\right).$$

Since we have surjectivity and  $CK_1$  is the same size as our target, all we have to do now is verify that the relations,

$$b_i^2 = 0 \quad \text{for } i < 2^{n-1} \quad \text{and} \quad b_{2^{n-1}+i}^2 = \sum_{k=0}^i u_n b_{2k} b_{2i-2k+1},$$

map to relations in  $K_*(\prod_i K_i)$ . In the range needed to define the map on the  $b_i$ ,  $K_*(\prod_i K_i)$  never has a product of two  $a_I$  with both having a nontrivial  $a_{(n-1)}$ . Consequently, every product in this range has square trivial because it must have a generator from degree less than  $2^{n-1}$  and which therefore has no  $a_{(n-1)}$  in it. Thus the only possibility for a non-trivial square is the  $a_I$  of the degree of  $b_i$ . In fact, the image of the  $b_i$  must contain this term. If  $a_I$  is primitive we know this already. If it is not primitive then it is hit because our coproduct determines our image of  $b_i$  and a lower  $b_j$  has hit the primitive associated with the coproduct of  $a_I$ . So, we need the relation associated with  $b_i^2$  to map to that for the square of  $a_I$  associated with the image of  $b_i$ . If  $i < 2^{n-1}$  then we know that all the  $a_I^2 = 0$  and also  $b_i^2 = 0$ . For  $2^{n-1} \leq i < 2^n$ , the square of  $a_I$  is, as we have already seen, a primitive in the same degree as the rest of the relation for  $b_i^2$ . Thus we know what element has to hit the primitive for our result to be true:

$$(3.15) \quad \sum_{k=0}^j b_{2k} b_{2j-2k+1}$$

has to hit the primitive in degree  $2(2j+1)$ . This is for  $b_{2^{n-1}+j}^2$  where  $j < 2^{n-1}$ . Since the degree of this is  $2(2j+1) < 2(2^n)$  we are in a range such that all squares of this degree are zero. If this element is primitive then we are done. When  $R = \mathbf{Z}[b_1, b_2, \dots]$  is given the structure of an Hopf algebra via:

$$\psi(b_n) = \sum_{i+j=n} b_i \otimes b_j.$$

The primitives are given by the Newton polynomials  $N_k(b_1, b_2, \dots, b_k)$ ,  $k > 0$  via the recursive definition

$$(3.16) \quad N_k = (-1)^{k-1} k b_k + \sum_{i=1}^{k-1} (-1)^{i-1} b_i N_{k-i}.$$

Define the following formal power series

$$\begin{aligned} \text{(i)} \quad N(t) &= \sum_{k=1}^{\infty} N_k t^k \\ \text{(ii)} \quad B(t) &= \sum_{k=0}^{\infty} (-1)^k b_k t^k \\ \text{(iii)} \quad A(t) &= \sum_{k=0}^{\infty} (-1)^k k b_k t^k \end{aligned}$$

Then notice that equation (3.16) may be rewritten as

$$(3.17) \quad N_k + \sum_{i=1}^{k-1} (-1)^i b_i N_{k-i} = -(-1)^k k b_k,$$

which is

$$(3.18) \quad N(t)B(t) = -A(t)$$

so we get

$$(3.19) \quad N(t) = -A(t)B(t)^{-1},$$

see [Ada74, page 94].

Let  $P_k$  be elements defined via  $\sum_{k=1}^{\infty} P_k t^k = P(t)$ , and

$$(3.20) \quad P(t) = -A(t)B(t)$$

In particular, we have:

$$P_q = (-1)^{q-1} q b_q + (-1)^{q-1} \sum_{i=1}^{q-1} i b_i b_{q-i}.$$

Modulo 2 this is just

$$P_q = q b_q + \sum_{i=1}^{q-1} i b_i b_{q-i}.$$

When  $q = 2j$  this, modulo 2, is  $\sum_{i=1}^{q-1} i b_i b_{2j-i}$  and our sum is only nontrivial when  $i$  is odd, so this reduces further to  $\sum_{k=1}^j b_{2k+1} b_{2j-2k-1}$ . This sum is symmetric and so terms pair up to be zero modulo 2 except possibly for a middle term squared which we already know maps to zero.

In the case where  $q$  is odd,  $q = 2j+1$ , we have

$$P_q = q b_q + \sum_{i=1}^{q-1} i b_i b_{q-i} = b_{2j+1} + \sum_{k=0}^{j-1} b_{2k+1} b_{2j-2k},$$

which can readily be seen to be our chosen element, Equation (3.15).

Now from 3.19 and 3.20, we get the equality  $P(t) = N(t)B(t)^2$ . Since  $B(t)^2$  maps to zero in our range, the proof follows.  $\square$

*Remark 3.21.* This isomorphism does not come from topology. There is certainly a map  $BO \rightarrow K(\mathbb{F}_2, 1) \times K(\mathbb{F}_2, 2)$  which is surjective for  $K_*(-)$ . The kernel of this map is realized by the Morava  $K$ -theory of the fibre  $BSpin$ , and there is no non-trivial map from this to  $K(\mathbb{F}_2, 3)$  because it is 3-connected. Likewise there can be no maps from the Eilenberg-Mac Lane spaces to  $BO$  because it would imply that  $K(\mathbb{F}_2, 2)$  splits off of  $BSO$ .

#### 4. HOPF ALGEBRAS AND THE BAR SPECTRAL SEQUENCE

We let  $h_*(-)$  be either Morava  $K$ -theory or standard mod  $p$  homology. In this paper we only use  $K(n)$  for  $p = 2$  but these preliminary results are true for the other theories so we state things in general. What is important for us is a Künneth isomorphism which we have for Morava  $K$ -theory and standard mod  $p$  homology. This gives us Hopf algebras. All of our spaces are connected infinite loop spaces and all of our maps are infinite loop maps. Consequently all of our theories always give us bicommutative Hopf algebras which have exhaustive primitive filtrations. Since we are using  $K(n)$  for  $p = 2$  we have to confirm we have commutativity for each of our spaces because this is not a commutative ring spectrum. This category of Hopf algebras is Abelian. In particular, all  $K(n)_*X$  with  $X$  a connected double loop space are exhaustive bicommutative Hopf algebras. For details on these Hopf algebras see [HRW98] and [SW98].

We need some results on this category.

**Theorem 4.1.** *Our category of bicommutative Hopf algebras with an exhaustive primitive filtration has the following properties:*

- (i) [Bou96a, Appendix] *The category is Abelian.*
- (ii) [Bou96b, Theorem B.7] *A sub-Hopf algebra of a polynomial Hopf algebra is polynomial too.*
- (iii) [Bou96b, Theorem B.9] *A short exact sequence of Hopf algebras which ends with a polynomial algebra is split as algebras.*

The Künneth isomorphism also gives us the bar spectral sequence. We rely heavily on the bar spectral sequence as a spectral sequence of Hopf algebras. This is discussed in depth in [HRW98, pages 144-5] and [RW80, pages 704-5].

**Theorem 4.2** (The Bar Spectral Sequence). *Let  $h_*(-)$  be Morava  $K$ -theory or standard mod  $p$  homology. Let*

$$F \xrightarrow{i} E \xrightarrow{r} B$$

*be a fibration of connected double loop spaces with  $h_*F$  a bicommutative Hopf algebra.*

- (i) *There is a spectral sequence of Hopf algebras:*

$$E_{*,*}^2 \simeq \mathrm{Tor}_{*,*}^{h_*F}(h_*E, h_*) \simeq \mathrm{Tor}_{*,*}^{\ker i_*}(h_*, h_*) \otimes \mathrm{coker} i_* \Rightarrow h_*B$$

*where  $\mathrm{coker} i_*$  is  $\mathrm{Tor}_{0,*}^{h_*F}(h_*E, h_*)$ .*

- (ii) *If  $i_*$  is injective then we have a short exact sequence of Hopf algebras:*

$$h_* \rightarrow h_*F \rightarrow h_*E \rightarrow h_*B \rightarrow h_*$$

(iii) With Restriction A (to follow) on our Hopf algebra  $\ker i_*$ , if  $\text{coker } i_*$  is even degree then  $\text{coker } i_*$  injects into  $h_*B$  and all differentials in the spectral sequence take place in  $\text{Tor}_{**}^{\ker i_*}(h_*, h_*)$ .

(iv) If

$$\Omega F \xrightarrow{\Omega i} \Omega E \xrightarrow{\Omega r} \Omega B$$

gives a short exact sequence of Hopf algebras

$$h_* \rightarrow h_*\Omega F \rightarrow h_*\Omega E \rightarrow h_*\Omega B \rightarrow h_*$$

with  $h_*\Omega B$  a Restriction A Hopf algebra and  $K_*F$  even degree, then we get a short exact sequence of Hopf algebras:

$$h_* \rightarrow h_*F \rightarrow h_*E \rightarrow h_*B \rightarrow h_*$$

*Remark 4.3.* We actually prove something stronger for part (iii) but don't need it in this paper. If you can write  $\text{coker } i_*$  as  $H \otimes E$  where  $H$  is an even degree Hopf algebra and  $E$  an exterior algebra on odd degree generators, then  $H$  injects into  $h_*B$ .

**Theorem 4.4.** Let  $K = K(n)$  ( $p$  arbitrary). Let  $E \rightarrow B' \rightarrow B$  be connective coverings of a simply connected double loop space  $B$ . Let  $F$  be the fibre of  $E \rightarrow B$  and  $F'$  the fibre of  $E \rightarrow B'$ . If

$$K_* \rightarrow K_*F \rightarrow K_*E \rightarrow K_*B \rightarrow K_*$$

is a short exact sequence of Hopf algebras then so is

$$K_* \rightarrow K_*F' \rightarrow K_*E \rightarrow K_*B' \rightarrow K_*.$$

*Proof.* We have maps of fibrations:

$$\begin{array}{ccccc} F' & \longrightarrow & E & \longrightarrow & B' \\ \downarrow & & \downarrow \simeq & & \downarrow \\ F & \longrightarrow & E & \longrightarrow & B. \end{array}$$

$F'$  and  $F$  are finite Postnikov systems and the homotopy of  $F'$  split injects into the homotopy of  $F$ . From [HRW98] we know this implies the Morava  $K$ -theory also injects. Thus  $K_*F'$  injects to  $K_*E$  and by Theorem 4.2 (ii) we have our result.  $\square$

*Remark 4.5. Restriction A.*

Hopf algebras over  $K(n)_*$  can get quite messy, [SW98]. To calculate the Tor we only need the algebra structure and  $\text{Tor}^{B \otimes C} \simeq \text{Tor}^B \otimes \text{Tor}^C$ . As algebras, only a few basic Hopf algebras show up in our work. We only allow tensor products of those listed below. In the theorem above, where the restriction applies, we consider only these Hopf algebras. For odd primes it is automatic that a Hopf algebra splits into the tensor product of an exterior Hopf algebra on odd degree generators and an even degree Hopf algebra. One of the consequences of our restriction here is that the same is true for  $p = 2$ . This is the only restriction on standard homology Hopf algebras.

- (i)  $P = h_*[x]$ , a polynomial algebra on an even degree generator. This is a restriction because for  $p = 2$  a polynomial algebra generator could have odd degree. This does not happen in our work so we make the same restriction for  $p = 2$  as Hopf algebras imply for odd primes.  $\text{Tor}^P \simeq E(\sigma x)$ , an

exterior algebra on an odd total degree element in the first filtration in the bar spectral sequence, i.e.  $\sigma x \in \text{Tor}_{1,|x|}^P$ .

- (ii)  $P_k = h_*[x]/(x^{p^k})$ , the truncated polynomial algebra on an even degree generator. Note the same restriction here as above. Also, when  $p = 2$  and  $k = 1$ ,  $h_*[x]/(x^2)$  is an exterior algebra. However, when the degree of  $x$  is even, we think of it as a truncated polynomial algebra and when the degree of  $x$  is odd we think of it as exterior.  $\text{Tor}^{P_k} \simeq E(\sigma x) \otimes \Gamma(\tau(x^{p^{k-1}}))$  with  $\sigma x \in \text{Tor}_{1,|x|}^{P_k}$  and  $\tau(x^{p^{k-1}}) \in \text{Tor}_{2,p^k|x|}^{P_k}$ . For  $p = 2$  and  $k = 1$  this is really  $\Gamma(\sigma x)$  but the little extra coalgebra structure does not change the algebra structure and so we will continue to think of it as an exterior algebra tensored with a divided power algebra.
- (iii) For  $h = K(n)$  we can have an even degree  $x$  with  $x^{p^k}$  written non-trivially in terms of lower powers of  $x$  in such a way that the Hopf algebra is irreducible, i.e. has no sub-Hopf algebras. These always have trivial Tor. See [SW98] for more details. The only living examples are in  $K(n)_*K(\mathbb{Z}/(p^k), n)$  and  $K(n)_*K(\mathbb{Z}, n+1)$  where we have  $x^p = \pm x$  (ignoring the powers of  $u_n$ ).
- (iv)  $E = E(x)$  an exterior algebra on an odd degree element. Again, this is a restriction, but only notational this time. Our even degree exterior generators are called truncated polynomial generators.  $\text{Tor}^E \simeq \Gamma(\sigma x)$  with  $\sigma x \in \text{Tor}_{1,|x|}^E$ .
- (v)  $P_\infty = \langle a_0, a_1, \dots \rangle$  with  $a_{k+1}^p = a_k$  and  $a_0^p = 0$ .  $K(n)_*K(\mathbb{Z}, q+1)$  when  $n > q$  is made up of these and  $\text{Tor}^{P_\infty} \simeq \Gamma(\tau(a_0))$  with  $\tau(a_0) \in \text{Tor}_{2,p|a_0|}^{P_\infty}$ .

For a divided power algebra,  $\Gamma(x)$  we refer to  $x$  as its “generator” even though it is not the algebra generator. It does serve as an adequate reference for the divided power algebra.

*Remark 4.6* (The differentials.). We need to discuss the behavior of differentials in the bar spectral sequence. Because the spectral sequence is as Hopf algebras we know that a differential must originate on a generator in filtration 2 or higher and hit a primitive ([Smi70, page 78]). All generators in filtration 2 or greater in Tor (for our Restriction A Hopf algebras) are  $\gamma_{p^i}(x)$  in some divided power algebra and they are in even degree. The targets must therefore be in odd degree and the only odd degree primitives are in either filtration 0 or 1. Thus each differential is computed on just one part of the total Tor, a  $E(x) \otimes \Gamma(y)$ , with  $d(\gamma_{p^i}(y)) = x$ . The resulting homology is the sub-Hopf algebra of  $\Gamma(y)$  generated by the  $\gamma_{p^j}(y)$  for  $j < i$ . Because of the length of this differential, these  $\gamma_{p^j}$ ,  $j < i$ , are permanent cycles. Furthermore, they can never be hit by differentials because they are even degree.

*Proof of Theorem 4.2.* We have already discussed the existence of the bar spectral sequence of part (i). However, the computation of the Tor is not so obvious. The source for this is [Smi67, Proposition 1.5]. He is working only with graded (connected) Hopf algebras. His proof depends on the theorem of [MM65] which says that a Hopf algebra is free over a sub-Hopf algebra. Their proof of this uses the grading but it is not necessary. Our Hopf algebras have nice coproducts and we can use what we call the primitive filtration to prove this same freeness result, see [Rad77]. For the map  $A \rightarrow B$  in the computation of  $\text{Tor}^A(B, k)$  we have that  $A$

is free over the kernel and that  $B$  is free over the image. Both are needed. Part (ii) follows immediately from the computation of Tor in part (i). Part (iii) follows from the above discussion about differentials. Differentials only hit odd exterior primitives and there are none of them in  $\text{coker } i_*$ , which is filtration 0. Part (iv) is a little different. Recall that  $K_*F$  is even degree. Let us look at the bar spectral sequence for  $\Omega B \rightarrow F \rightarrow E$ . Since  $K_*\Omega E$  surjects to  $K_*\Omega B$ , we know that  $K_*\Omega B \rightarrow K_*F$  is trivial, so the cokernel term,  $K_*F$ , injects to  $K_*E$  by (iii). Then, by part (ii) we get our short exact sequence.  $\square$

We need the generalized Atiyah-Hirzebruch spectral sequence. See, for example, [Dye69, pages 24–25]. We will use it in a nontrivial way.

**Theorem 4.7.** *Let  $h_*(-)$  be a generalized homology theory. For a fibration  $F \rightarrow E \rightarrow B$ , there is a spectral sequence*

$$E^2 \simeq H_*(B, h_*F) \Rightarrow h_*E.$$

From [RWY98, Proposition 2.0.1, page 155] we have:

**Theorem 4.8.** *Let  $K = K(n)$ . Let  $F \rightarrow E \rightarrow B$  be a fibration.*

- (i) *If  $K_*F$  is even degree and  $H_*(B; \mathbb{F}_p)$  is even degree, then the generalized Atiyah-Hirzebruch spectral sequence collapses.*
- (ii) *If the fibration is one of double loop spaces then we have a short exact sequence of Hopf algebras:*

$$K_* \rightarrow K_*F \rightarrow K_*E \rightarrow K_*B \rightarrow K_*.$$

We need the following theorem:

**Theorem 4.9.** *The homologies of  $\underline{bo}_2$  and  $\underline{bo}_4$  are polynomial on even degree generators and the cohomology of  $\underline{bo}_4$  is also polynomial.*

*Proof.* For a contemporary reference for  $\underline{bo}_4 = BSp$  see [Koc96].  $H_*\underline{bo}_2$  follows quickly from a double application of the Eilenberg-Moore spectral sequence.  $\square$

*Remark 4.10.* We need some basic facts about the Morava  $K$ -theory of Eilenberg-Mac Lane spaces from [RW80]. First,  $K_*K(\mathbb{F}_p, i)$  is a finite Hopf algebra concentrated in even degrees. It is trivial if  $i > n$  so that Morava  $K$ -theory only sees a finite number of Eilenberg-Mac Lane spaces.  $K_*K(\mathbb{Z}, i)$  is even degree and infinite and the bar spectral sequence converging to  $K_*K(\mathbb{Z}, i+1)$  collapses. It is trivial if  $i > n$ . There is a short exact sequence of Hopf algebras:

$$(4.11) \quad K_* \twoheadrightarrow K_*K(\mathbb{F}_p, i) \twoheadrightarrow K_*K(\mathbb{Z}, i+1) \xrightarrow{P_*} K_*K(\mathbb{Z}, i+1) \twoheadrightarrow K_*.$$

The Morava  $K$ -theory of Eilenberg-Mac Lane spaces always satisfies Restriction A.

## 5. MAIN CALCULATION

**5.1. Introduction.** In this section we will prove the theorems of Section 2. Our organization is somewhat different here than in that section. There, the results are stated for each Omega spectrum separately. Here, we are inclined to go up the connective covers a bit and then deloop and do it again. After awhile, we can do things in bulk and prove Theorem 1.3. Then things sort of get very *ad hoc*. This is not the historical way things were done. That would be much too confusing. Nor do we start with the known results and work from there, or, do the easiest first and move on. We do try to be as systematic as we find possible though.

**5.2. Bicommutativity.** We work exclusively with  $p = 2$  and a minor problem presents itself. The spectra  $K(n)$  for  $p = 2$  are not commutative ring spectra but we rely heavily not just on Hopf algebra commutativity, but Hopf algebra cocommutativity as well. Without it we cannot compute our Tors and we wouldn't have an abelian category to work in. Fortunately, the obstruction to commutativity is well understood, [Wür86, pages 36–37] and [Mir79]. Let  $m$  be the multiplicative map and  $T$  the switch map, then

$$m \circ T = m + u_n m(Q_{n-1} \wedge Q_{n-1})$$

where  $Q_{n-1}$  is a stable operation of degree  $2^n - 1$  (negative when acting on homology). All we need here is that it is an odd degree stable operation. We use this to show commutativity. In particular, if we compute that something is even degree then it is commutative. We will have to verify commutativity for every space we consider. Rather than have a separate section we will put in [square brackets] all our comments as we prove our results in this section.

We remind the reader that  $RP^\infty = K(\mathbb{F}_2, 1)$  and  $CP^\infty = K(\mathbb{Z}, 2)$ . We will use them interchangeably.

**5.3.  $\mathbf{X}_0$ .** We start with  $\underline{b}O_0 = \mathbb{Z} \times BO$  where we have  $BO = \underline{B}O_0$ , so getting  $K_*\underline{B}O_0$  really starts two of our sequences. This is done explicitly in [RWY98, Section 2.5, p. 161] using the Atiyah-Hirzebruch spectral sequence, see equations (3.1) and (3.2). There it is shown to be a polynomial algebra. Next, we know that  $B\mathcal{S}O \times RP^\infty \simeq BO$ , or, rephrased,  $\underline{B}O_0 \simeq \underline{B}\mathcal{S}O_0 \times RP^\infty$ . This gives a short exact sequence of Hopf algebras,

$$K_* \rightarrow K_*\underline{B}\mathcal{S}O_0 \rightarrow K_*\underline{B}O_0 \rightarrow K_*RP^\infty \rightarrow K_*$$

A sub-Hopf algebra of a polynomial Hopf algebra is polynomial too, see Theorem 4.1. Thus we have that  $K_*\underline{B}\mathcal{S}O_0$  is a polynomial sub-Hopf algebra of  $K_*\underline{B}O_0$ .

Next, comes  $K_*BSpin$  where  $BSpin = \underline{B}Spin_0$ . We have a sequence of fibrations:

$$Spin \rightarrow SO \rightarrow RP^\infty \rightarrow BSpin \rightarrow BSO \rightarrow \dots$$

Because  $RP^\infty$  splits off of  $SO$  we know the map  $K_*RP^\infty \rightarrow K_*BSpin$  is trivial. Thus, when we use the bar spectral sequence to compute  $K_*BSO$  the  $E^2$  term is  $K_*BSpin \otimes \text{Tor}^{K_*RP^\infty}$ . We know that this converges to  $K_*BSO$  which we know is polynomial (on even degree generators) and we want to know the same is true for  $K_*BSpin$ .  $K_*BSpin$  sits in the zeroth filtration of the bar spectral sequence. We know the Tor part completely from [RW80]. The Tor part consists of some exterior generators ( $n - 1$  of them in filtration one) and the same number of divided power Hopf algebras ( $n - 1$  of them starting in filtration 2). Thus, in the spectral sequence, all differentials must originate on this divided power algebra part and hit exterior generators. Since there is just enough divided power algebra stuff to kill off the exterior generators in Tor, and all exterior generators must be hit, then we can conclude that is precisely what happens. Consequently we can see that  $K_*BSpin$  injects into  $K_*BSO$  and is even degree and by Theorem 4.2 (ii)

$$BSpin \rightarrow BSO \rightarrow K(\mathbb{F}_2, 2)$$

gives rise to a short exact sequence in Morava  $K$ -theory so  $K_*BSpin$  is a sub polynomial Hopf algebra of  $K_*BSO$ .

We have now done the zeroth spaces in our four Omega spectra. [Since all are even degree, all are bicommutative.]

5.4.  $\underline{X}_1$ . We can now move on to the  $\underline{X}_1$  for these spectra. By the bar spectral sequence we get all are exterior algebras on odd degree generators. [The generators are all suspensions of generators from  $\underline{X}_0$  which had the stable operation associated with lack of commutativity zero. The trivial obstruction suspends to zero as well so these, too, are bicommutative.]

We want the maps too. Since  $\underline{bo}_0 \simeq \mathbb{Z} \times \underline{BO}_0$  we have  $\underline{bo}_1 \simeq S^1 \times \underline{BO}_1$ . Since  $K_*\underline{BO}_0$  is polynomial, both of these are exterior and we get the split short exact sequence of Hopf algebras:

$$K_* \rightarrow K_*\underline{BO}_1 \rightarrow K_*\underline{bo}_1 \rightarrow K_*S^1 \rightarrow K_*.$$

Next we study the fibrations

$$RP^\infty \rightarrow \underline{BSO}_1 \rightarrow \underline{BO}_1 \rightarrow K(\mathbb{F}_2, 2).$$

We know that  $K_*\underline{BSO}_1$  and  $K_*\underline{BO}_1$  are exterior from the bar spectral sequence.  $K_*RP^\infty$  is even degree so the first map is trivial in Morava  $K$ -theory. The bar spectral sequence converging to  $K_*\underline{BO}_1$  thus has  $E^2$  term  $K_*\underline{BSO}_1 \otimes \text{Tor}^{K_*RP^\infty}$ . The Tor part is well understood. It has an exterior part and a divided power algebra part. There are the same number of exterior generators ( $n-1$  in filtration 1) as there are divided power towers (starting in filtration 2). Since we can have no even degree generators when we are done, the only thing which can happen is that the divided power tower generators in filtration 2 have differentials to exterior generators in filtration 0 (the differentials have to kill all the even stuff and must be at least a  $d_2$ ). This tells us exactly how many exterior generators are in the kernel, ( $n-1$ ), and cokernel of the map  $K_*\underline{BSO}_1 \rightarrow K_*\underline{BO}_1$ , ( $n-1$ ).

Next we study the fibrations

$$K(\mathbb{F}_2, 2) \rightarrow \underline{BSpin}_1 \rightarrow \underline{BSO}_1 \rightarrow K(\mathbb{F}_2, 3).$$

The same thing happens here! We know the two middle terms are exterior and the left one is even degree. So, the argument proceeds just as in the previous case and we know exactly how many,  $\binom{n-1}{2}$ , exterior generators of  $K_*\underline{BSpin}_1$  are in the kernel of the map to  $K_*\underline{BSO}_1$  and how many are in the cokernel (the same number).

5.5.  $\underline{X}_2$ . Since  $K_*\underline{X}_1$  is exterior on odd degree generators we know that the bar spectral sequence collapses as divided power algebras all in even degrees [and are therefore bicommutative]. However, getting the algebra structure and the maps is not so easy. First, we look at  $\underline{bo}_2$ . The standard homology is easy to compute and it is polynomial on even degree generators, Theorem 4.9. Thus we see, by the Atiyah-Hirzebruch spectral sequence, that  $K_*\underline{bo}_2$  is polynomial on even degree generators. For  $\underline{BO}_2$  we look at the fibrations

$$\underline{BO}_1 \rightarrow \underline{bo}_1 \rightarrow S^1.$$

We know this is short exact and split as Hopf algebras. Thus the Tor groups in the 3 bar spectral sequences are also short exact. Since they are all even degree the spectral sequences all collapse and give rise to the short exact sequence of Hopf algebras:

$$K_* \rightarrow K_*\underline{BO}_2 \rightarrow K_*\underline{bo}_2 \rightarrow K_*CP^\infty \rightarrow K_*.$$

Since we know the middle term is polynomial we have the left side is also polynomial because it is a sub-Hopf algebra of a polynomial algebra (recall Theorem 4.1).

Our style changes dramatically here. At this point we can go from proving things one space at a time to getting results in bulk. We have to start this part of our study by looking at the fibration

$$S^1 \rightarrow \underline{BO}_2 \rightarrow \underline{bo}_2.$$

Keep in mind that we already know the Morava  $K$ -theory for these spaces. The  $E^2$  term of the generalized Atiyah-Hirzebruch spectral sequence for the fibration is

$$H_*(\underline{bo}_2; K_*S^1).$$

We know that all of the odd stuff has to go away. There can be no differentials on the fibre stuff, i.e.  $K_*S^1$ , so it must be the target of a differential which comes from  $H_*(\underline{bo}_2; \mathbb{F}_2)$ . We now compare this spectral sequence with another. Let  $Y \rightarrow \underline{bo}_2$  be a connective cover. Since we know the homotopy of  $\underline{bo}_2$  we can easily compute the homotopy of the fibre  $F$ . We see that  $F$  has a circle which splits off of it. Write  $F = F' \times S^1$ . Then we have maps of fibrations

$$\begin{array}{ccccc} F & \rightarrow & Y & \rightarrow & \underline{bo}_2 \\ \downarrow & & \downarrow & & \parallel \\ S^1 & \rightarrow & \underline{BO}_2 & \rightarrow & \underline{bo}_2 \end{array}$$

where  $Y \rightarrow \underline{BO}_2$  is also a connective cover. The generalized Atiyah-Hirzebruch spectral sequence maps, on  $E^2$  terms, as follows:

$$\begin{array}{ccc} H_*(\underline{bo}_2; K_*F) & \Longrightarrow & K_*Y \\ \downarrow & & \downarrow \\ H_*(\underline{bo}_2; K_*S^1) & \Longrightarrow & K_*\underline{BO}_2 \end{array}$$

Because  $S^1$  splits off of  $F$ , the top  $E^2$  term is

$$H_*(\underline{bo}_2; K_*F') \otimes H_*(\underline{bo}_2; K_*S^1).$$

The first term is all even degree and the spectral sequence for  $\underline{BO}_2$  forces all the differentials in the right hand side. Then everything is in even degrees and we can see how the fibration  $F' \rightarrow Y \rightarrow \underline{BO}_2$  behaves under Morava  $K$ -theory. We see from the spectral sequence that  $K_*F'$  injects into  $K_*Y$  and from this and Theorem 4.2 (ii) we get a short exact sequence:

$$K_* \rightarrow K_*F' \rightarrow K_*Y \rightarrow K_*\underline{BO}_2 \rightarrow K_*.$$

Furthermore, since the right hand term is polynomial, these all split as algebras. It is important to note that we know all about  $K_*F'$  as well.

This concludes our calculation of all  $Y$  which are connective covers of  $\underline{BO}_2$ . We can now write these down in our usual notation. They are:  $\underline{BSO}_{2+8i}$ ,  $\underline{BSpin}_{2+8i}$ ,  $\underline{bo}_{10+8i}$ , and  $\underline{BO}_{10+8i}$ , where  $i \geq 0$ . [They are all even degree and so bicommutative.] By Theorem 4.4 we get much more of our Theorem 1.3, namely, the  $i = 2$  modulo 8 part.

**5.6.  $\underline{X}_3$ .** We are now ready to compute the deloopings of all of these spaces to get  $\underline{X}_i$  where  $i = 3$  modulo 8 now that we know all of these for  $i = 2$  modulo 8.

We start with the fibration

$$CP^\infty \rightarrow \underline{BO}_3 \rightarrow \underline{bo}_3.$$

We know that  $K_*\underline{BO}_2$  and  $K_*\underline{bo}_2$  are polynomial so we know  $K_*\underline{BO}_3$  and  $K_*\underline{bo}_3$  are exterior [and because the generators are suspensions of generators with trivial obstructions, these too are bicommutative]. We just need to study the map between

them. Since  $K_*CP^\infty$  is all even degree the first map must be trivial and so the bar spectral sequence for the above fibration is just  $K_*\underline{BO}_3 \otimes \text{Tor}^{K_*CP^\infty}$  and we know the Tor term completely from [RW80] or Restriction A. It is a divided power algebra with primitives in filtration 2. Since our answer is  $K_*\underline{bo}_3$  which is exterior, this must all go away. The only way this can happen is if there are differentials from the filtration 2 primitives to exterior generators in filtration 0. We know exactly how many such elements there are,  $(n-1)$ . Surprisingly, we see that the remaining answer, after these differentials, is all in filtration zero and so we have a surjection:  $K_*\underline{BO}_3 \rightarrow K_*\underline{bo}_3$  and we know how many exterior generators there are in the kernel,  $(n-1)$ . We had to do this space separately, but now we can move on to once again do things in bulk.

Let us recall the fibration  $F' \rightarrow Y \rightarrow \underline{BO}_2$ . The conditions of Theorem 4.2 (iv) are met so we get a short exact sequence of Hopf algebras (with  $B(-)$  the delooping):

$$K_* \rightarrow K_*BF' \rightarrow K_*BY \rightarrow K_*\underline{BO}_3 \rightarrow K_*.$$

[Although  $K_*\underline{BO}_3$  is exterior and bicommutative there is still some possibility that the obstructions to bicommutativity for  $K_*BY$  lie in the kernel here, i.e.  $K_*BF'$ . However, these exterior generators in  $K_*BY$  are suspensions of generators in  $K_*Y$  which have trivial obstructions on them, so they are trivial in  $K_*BY$  too and we have bicommutativity.] We again use Theorem 4.4 to prove more of Theorem 1.3, the  $i = 3$  modulo 8 part.

5.7. **X<sub>4</sub>**. We now proceed to do the  $\underline{X}_i$ ,  $i = 4$  modulo 8 cases. This turns out to be the easiest case of all. In the case of  $\underline{bo}_4$  we know that there is no torsion in regular homology and that it is polynomial, Theorem 4.9. Thus the Atiyah-Hirzebruch spectral sequence collapses and it too is polynomial. We can then take the connective covers,  $Y$ , and fibre,  $F$ , (a finite Postnikov system with even Morava  $K$ -theory), and we have a short exact sequence of Hopf algebras

$$K_* \rightarrow K_*F \rightarrow K_*Y \rightarrow K_*\underline{bo}_4 \rightarrow K_*$$

from Theorem 4.8 which is split as algebras because the right hand one is polynomial. This completes the calculation of all  $K_*\underline{X}_i$  for  $i = 4$  modulo 8 in our 4 Omega spectra. [Everything is even degree so we get bicommutativity.] Just as an example of something new here, we have not computed  $K_*\underline{BO}_4$  by itself. However, it follows from this result that the fibration:

$$K(\mathbb{Z}, 3) \rightarrow \underline{BO}_4 \rightarrow \underline{bo}_4$$

gives us a short exact sequence which is split as algebras, so, as algebras,

$$K_*\underline{BO}_4 \simeq K_*K(\mathbb{Z}, 3) \otimes K_*\underline{bo}_4.$$

More of Theorem 1.3 follows from Theorem 4.4, the  $i = 4$  modulo 8 part.

5.8. **Theorem 1.3.** Once we prove Theorem 1.3 for  $\underline{bo}_i$  and show that  $K_*\underline{bo}_i$  satisfies Restriction A, we get Theorem 1.3 for  $\underline{bo}_{i+1}$  and then Theorem 4.4 gives it for  $i$  and  $i+1$  modulo 8. Thus, all we need to complete the proof is to calculate  $K_*\underline{bo}_i$  and show it meets Restriction A for  $i = 5, 6, 7$  and 8. For this result we don't need it, but of course we want  $i = 9$  for completeness.

5.9. **X<sub>5</sub>**. The bar spectral sequence from  $K_*\underline{bo}_4$  to  $K_*\underline{bo}_5$  is exterior and collapses. [It is bicommutative because the generators are the suspensions of generators with trivial obstructions on them. For a connective cover,  $Y$ , the only possibility is that the obstructions on the exterior generators lands in  $K_*F$  but those exterior generators are also suspensions of elements with trivial obstructions, so  $K_*Y$  is bicommutative.]

5.10. **Review**. Let us consolidate our gains now. We have computed the Morava  $K$ -theory of  $\underline{bo}_i$  and all its connective covers for  $i = 2, 3, 4$  and  $5$ , plus  $\underline{X}_i$  for  $i = 0$  and  $1$ . We are left with the necessity to compute  $K_*\underline{bo}_6$ ,  $K_*\underline{bo}_7$ ,  $K_*\underline{bo}_8$ , and  $K_*\underline{bo}_9$ . Their connective covers are automatic from the preceding subsections. All other spaces have been computed except for some of the negatively indexed spaces:  $\underline{BO}_{-1} = O$ ,  $\underline{BSO}_{-1} = SO$ ,  $\underline{BSO}_{-2} = O/U$ ,  $\underline{BSpin}_{-1} = Spin$ ,  $\underline{BSpin}_{-2} = SO/U$ ,  $K_*\underline{BSpin}_{-3} \simeq S^1 \times \underline{bo}_5$  and  $K_*\underline{BSpin}_{-4} \simeq \mathbb{Z} \times \underline{bo}_4$ . These last two we can see the answers for immediately in terms of things we already know. Also,  $\underline{BO}_{-1}$  is  $\mathbb{F}_2 \times SO$  and  $\underline{BSO}_{-2}$  is  $\mathbb{F}_2 \times \underline{BSpin}_{-2}$ .

This cuts our list of unknowns down to  $\underline{bo}_6$ ,  $\underline{bo}_7$ ,  $\underline{bo}_8$ ,  $\underline{bo}_9$ ,  $\underline{BSO}_{-1} = SO$ ,  $\underline{BSpin}_{-1} = Spin$ , and  $\underline{BSpin}_{-2} = SO/U$ . In addition to this list, there are just a few maps left unstudied. The ones we do not know yet are:  $Spin \rightarrow SO$ ,  $\underline{bo}_6 \rightarrow \underline{BSpin}_{-2}$ ,  $\underline{bo}_7 \rightarrow Spin$ ,  $\underline{bo}_8 \rightarrow \underline{BSpin}_{-1}$ ,  $\underline{bo}_9 \rightarrow \underline{BSpin}_{-1}$ .

So far, in our proofs, we have not yet had to use the spaces  $\underline{bu}_*$ . This is about to change.

5.11. **SO**. We have already computed the cokernel of  $K_*BU \rightarrow K_*BO$  as  $CK_1$ . We can use this in the spectral sequence for the fibration  $BU \rightarrow BO \rightarrow SO$ . The kernel of  $K_*BU \rightarrow K_*BO$  is known already to be polynomial on generators with lead terms the odd  $b_{2q+1}$ . We compare the spectral sequences for the two fibrations:

$$(5.11.1) \quad \begin{array}{ccccc} BU & \longrightarrow & BO & \longrightarrow & SO \\ \simeq \downarrow & & \downarrow & & \downarrow \\ BU & \longrightarrow & * & \longrightarrow & SU. \end{array}$$

Since the kernel is polynomial we see that for  $SO$  the spectral sequence collapses to  $CK_1 \otimes E$ .  $CK_1$  is the cokernel in filtration 0 and  $E$  is Tor of the kernel which is polynomial and so the exterior generators are in filtration one and have trivial differentials. We see that  $E$  injects to  $K_*SU$ . [Bicommutativity is harder to come by in this case than in previous ones. As we have computed it, it is possible that  $Q_{n-1}$  of an exterior generator is nonzero in  $CK_1$ . To solve this problem we use the fibration  $\underline{BSpin}_{-3} \rightarrow U \rightarrow SO$ .  $\underline{BSpin}_{-3} \simeq \underline{bo}_5 \times S^1$ . The Morava  $K$ -theory of this has already been shown to be exterior on odd degree generators and bicommutative. We know the same is true for  $U$  (the generators are in the image of the even degree generators of  $\underline{bu}_0$ ). Using the bar spectral sequence for this fibration we see that both the kernel and the cokernel must be exterior algebras. Our Tor is thus an exterior algebra, the cokernel, coming from  $K_*U$  in filtration zero, and Tor of the kernel, an even degree divided power algebra. There could be differentials but we don't care. All we care about is that all of the exterior generators of  $K_*SO$  come from  $K_*U$  and therefore the obstructions to commutativity on them are all trivial.]

5.12. **Spin.** We know that  $BSO \rightarrow BO \rightarrow K(\mathbb{F}_2, 1)$  gives a short exact sequence so we know that the cokernel of  $K_*BU \rightarrow K_*BSO$  is  $CK_2$ . We can now use the fibration  $BU \rightarrow BSO \rightarrow Spin$  to compute  $K_*Spin$  in the same way we computed  $K_*SO$ . From this the fibration  $Spin \rightarrow SO \rightarrow K(\mathbb{F}_2, 1)$  easily reduces to a short exact sequence:

$$(5.12.1) \quad K_* \rightarrow CK_2 \otimes E \rightarrow CK_1 \otimes E \rightarrow K_*K(\mathbb{F}_2, 1) \rightarrow K_*.$$

[Bicommutativity follows from the injection.]

5.13.  **$\underline{bo}_6$ .** We study the fibration  $\underline{bu}_3 \rightarrow Spin \rightarrow \underline{bo}_6$ .  $K_*\underline{bu}_3$  is exterior and hits all the exterior generators of  $K_*Spin$  but does not hit any of  $CK_2$ . The kernel must be exterior so Tor is  $CK_2$  tensored with a divided power algebra generated by the suspensions of the exterior generators in the kernel. This is all even degree so collapses. [And we have bicommutativity.] Comparing fibrations:

$$(5.13.1) \quad \begin{array}{ccccc} \underline{bu}_3 & \longrightarrow & Spin & \longrightarrow & \underline{bo}_6 \\ \simeq \downarrow & & \downarrow & & \downarrow \\ \underline{bu}_3 & \longrightarrow & * & \longrightarrow & \underline{bu}_4 \end{array}$$

we see the divided power part must inject into  $K_*\underline{bu}_4$  which is polynomial. So,  $K_*\underline{bo}_6$  is  $CK_2 \otimes P$ . Furthermore, the polynomial part splits off of  $K_*\underline{bu}_4$  as algebras. This is more difficult to see. Consider the fibration  $\underline{bo}_6 \rightarrow \underline{bu}_4 \rightarrow BSpin$ .  $K_*\underline{bu}_4$  is polynomial so the cokernel,  $K_*\underline{bu}_4//P$  is even degree and the kernel,  $CK_2$ , is Restriction A. So,  $K_*\underline{bu}_4//P$  injects into  $K_*BSpin$  by Theorem 4.2 (iii). Since this is polynomial,  $K_*\underline{bu}_4//P$  is polynomial so  $K_*\underline{bu}_4$  splits as these algebras.

5.14.  **$\underline{BSpin}_{-2}$ .** From the splitting  $\underline{BSpin}_{-3} \simeq S^1 \times \underline{bo}_5$  and the spectral sequences for the delooping of these spaces, we get evenly graded spectral sequences which must collapse and be short exact [giving bicommutativity]:

$$K_* \rightarrow K_*\underline{bo}_6 \rightarrow K_*\underline{BSpin}_{-2} \rightarrow K_*CP^\infty \rightarrow K_*.$$

But this does nothing obvious to elucidate the algebra structure of  $K_*\underline{BSpin}_{-2}$ . We use the fibration:

$$SO \rightarrow \underline{BSpin}_{-2} \rightarrow \underline{bu}_2.$$

We need the maps:

$$\begin{array}{ccccc} SO & \longrightarrow & \underline{BSpin}_{-2} & \longrightarrow & \underline{bu}_2 \\ \downarrow & & \downarrow & & \downarrow \\ RP^\infty & \longrightarrow & CP^\infty & \xrightarrow{2} & CP^\infty. \end{array}$$

All the vertical maps are surjections in Morava  $K$ -theory. We know from the injection  $K_*\underline{bo}_6 \rightarrow \underline{BSpin}_{-2}$  that we have a finite subalgebra  $CK_2$ . Using the bar spectral sequence for  $SO \rightarrow \underline{BSpin}_{-2} \rightarrow \underline{bu}_2$  we know  $CK_2$  must not be in the cokernel because the cokernel is even degree and so injects to the polynomial algebra  $K_*\underline{bu}_2$  by Theorem 4.2 (iii). So this coker is polynomial also. However,  $CK_2$  can be hit by  $K_*SO$  so it must be. Likewise, the  $K_*RP^\infty$  part of  $K_*SO$  is

carried along by the above diagram. Thus we see that  $CK_1$  injects from  $K_*SO$  to  $K_*\underline{BSpin}_{-2}$  and the cokernel is polynomial. We see that

$$K_*\underline{BSpin}_{-2} \simeq CK_1 \otimes P.$$

Note how we yet again use the spectral sequence starting with the thing we don't know and without computing the spectral sequence we end up knowing it.

5.15. **Review again.** Our list of unknowns now stands at  $K_*\underline{bo}_7$ ,  $K_*\underline{bo}_8$  and  $K_*\underline{bo}_9$ . The maps are:  $\underline{bo}_7 \rightarrow Spin$ ,  $\underline{bo}_8 \rightarrow BSpin$ , and  $\underline{bo}_9 \rightarrow \underline{BSpin}_1$ .

5.16.  **$\underline{bo}_7$ .** Consider the diagram

$$\begin{array}{ccc} K_*BSO & \longrightarrow & K_*Spin \\ \downarrow & & \downarrow \\ K_*K(\mathbb{F}_2, 2) & \longrightarrow & K_*K(\mathbb{Z}, 3). \end{array}$$

The left vertical map is surjective. The image of the top horizontal map is  $CK_2$ . The lower horizontal map injects so we get a surjection  $CK_2 \rightarrow K_*K(\mathbb{F}_2, 2) \subset K_*K(\mathbb{Z}, 3)$ .

Because  $K_*\underline{BSpin}_{-2}$  surjects to  $K_*CP^\infty$  we have  $K_*CP^\infty$  maps trivially to  $K_*\underline{bo}_7$ . So, we compare bar spectral sequences for the fibrations:

$$\begin{array}{ccccc} CP^\infty & \longrightarrow & \underline{bo}_7 & \longrightarrow & Spin \\ \downarrow & & \downarrow & & \downarrow \\ CP^\infty & \longrightarrow & * & \longrightarrow & K(\mathbb{Z}, 3) \end{array}$$

We know the bottom spectral sequence from [RW80] or Restriction A. Tor is a divided power algebra on  $n-1$  “generators” in filtration 2. Tor in the top one is just  $K_*\underline{bo}_7$  (the cokernel in filtration zero) and  $\text{Tor}^{K_*CP^\infty}$ . Since we know the image of the right vertical map we know which elements must be the source of differentials and that they must kill off exterior generators (and how many  $(n-1)$ ) in  $K_*\underline{bo}_7$ . Since this converges to  $K_*Spin$  which we know to be exterior and  $CK_2$ , then  $K_*\underline{bo}_7$  must be exterior tensor with  $CK_3$ . It is important to note that there are some,  $(n-1)$ , exterior generators in the kernel of the map  $K_*\underline{bo}_7 \rightarrow K_*Spin$  and that all of the exterior generators of  $K_*Spin$  are in the image of this map. [Bicommutativity isn't obvious. We could have our odd degree operation on something in  $CK_3$  ending on one of the exterior generators in the kernel. To solve this problem we use the fibration  $BSU \rightarrow BSpin \rightarrow \underline{bo}_7$ . The cokernel part of Tor is even degree and the kernel must be polynomial so Tor of it is exterior and we see that the spectral sequence collapses. Since the even degree stuff is in filtration zero and the odd generators are in filtration one, we cannot have the above possibility occur so we have bicommutativity.]

5.17.  **$\underline{bo}_8$ .** Consider the fibration

$$Spin \rightarrow K(\mathbb{Z}, 3) \rightarrow \underline{bo}_8.$$

We know the cokernel of the first map is just another  $K_*K(\mathbb{Z}, 3)$  from Remark 4.10 and the previous subsection. Since it is even degree it injects into  $K_*\underline{bo}_8$  by Theorem 4.2 (iii). This gives us the  $K(\mathbb{F}_2, 2) \rightarrow K(\mathbb{Z}, 3) \rightarrow \underline{bo}_8$  part of Theorem 1.5.

Now consider the fibration

$$K(\mathbb{Z}, 3) \rightarrow \underline{bo}_8 \rightarrow BSpin.$$

We know what the image and kernel of the first map are, and we know what the spectral sequence converges to. Since the kernel is  $K_*K(\mathbb{F}_2, 2)$  we know Tor for this. It has, as usual, a bunch of exterior generators in filtration 1 and a bunch of divided power algebras with primitives in filtration 2. Filtration 0 is  $K_*\underline{bo}_8$  modulo the image of  $K_*K(\mathbb{Z}, 3)$ , which is another copy of  $K_*K(\mathbb{Z}, 3)$ . Since  $K_*BSpin$  is polynomial, there can be no exterior elements left in the spectral sequence when we are done. The number of exterior generators in filtration one is precisely the same as the number of possible sources for differentials. Thus there can be no sources left over to hit anything in filtration zero. Thus  $K_*\underline{bo}_8 // K_*K(\mathbb{Z}, 3)$  is polynomial since it injects to the polynomial  $K_*BSpin$ . So  $K_*\underline{bo}_8$  is  $K_*K(\mathbb{Z}, 3)$  tensor with a polynomial algebra as algebras [and so bicommutative]. We cannot obviously determine the cokernel of the map of  $\underline{bo}_8$  to  $BSpin$  from this but we want it to be  $K_*K(\mathbb{F}_2, 3)$ . Certainly it is even degree and so injects into  $K_*K(\mathbb{Z}, 4)$ . We have an array of fibrations (both vertical and horizontal):

$$\begin{array}{ccccc} \underline{BSpin}_1 & \longrightarrow & \underline{bo}_8 & \longrightarrow & \underline{bu}_6 \\ \downarrow & & \downarrow & & \downarrow \\ \underline{BSO}_1 & \longrightarrow & BSpin & \longrightarrow & \underline{bu}_4 \\ \downarrow & & \downarrow & & \downarrow \\ K(\mathbb{F}_2, 3) & \longrightarrow & K(\mathbb{Z}, 4) & \xrightarrow{2} & K(\mathbb{Z}, 4) \end{array}$$

The right top vertical map  $K_*\underline{bu}_6 \rightarrow K_*\underline{bu}_4$  is known to be surjective so the right bottom vertical map  $K_*\underline{bu}_4 \rightarrow K_*K(\mathbb{Z}, 4)$  is trivial. Thus, the map from  $K_*BSpin$  to the lower right hand corner must be zero since it factors through  $K_*\underline{bu}_4$ . Going the other way, we see that the map  $K_*BSpin \rightarrow K_*K(\mathbb{Z}, 4)$  must factor through  $K_*K(\mathbb{F}_2, 3)$  because we go to zero in the right lower corner and we have the short exact sequence (and we are working in an abelian category):

$$K_* \longrightarrow K_*K(\mathbb{F}_2, 3) \longrightarrow K_*K(\mathbb{Z}, 4) \xrightarrow{2_*} K_*K(\mathbb{Z}, 4) \longrightarrow K_*.$$

We need the additional array of commuting maps and fibrations.

$$\begin{array}{ccccccc} & & & & & & BSpin \\ & & & & & & \downarrow \\ & & & & & & K(\mathbb{F}_2, 3) \rightarrow K(\mathbb{Z}, 4) \\ & & & & & & \downarrow \\ \underline{BSpin}_1 & \rightarrow & \underline{bo}_8 & \rightarrow & \underline{bu}_6 & \rightarrow & \underline{BSpin}_2 \rightarrow \underline{bo}_9 \end{array}$$

First we look at the bar spectral sequence for the vertical fibration on the right. Whatever the image of the top map is, call it  $L$ , we have Tor zero is  $K_*K(\mathbb{Z}, 4)/L$  which is all in even degrees and so injects into  $K_*\underline{bo}_9$ . We know that  $L$  factors through  $K_*K(\mathbb{F}_2, 3)$  so if we can show that  $K_*K(\mathbb{F}_2, 3)$  goes to zero in  $K_*\underline{bo}_9$  then we have it is equal to  $L$ . By the commuting diagram,  $K_*K(\mathbb{F}_2, 3)$  to  $K_*\underline{bo}_9$  factors through  $K_*\underline{BSpin}_2$  so if we can show that the map  $K_*\underline{BSpin}_2$  to  $K_*\underline{bo}_9$  is trivial then we are done. This will be trivial if the map of  $K_*\underline{bu}_6$  to  $K_*\underline{BSpin}_2$  is surjective and this will happen if the map  $K_*\underline{bo}_8$  to  $K_*\underline{bu}_6$  is injective (because the

bar spectral sequence for  $K_*\underline{BSpin}_2$  will collapse and give a short exact sequence). To see this injection, we look at the bar spectral sequence for the fibration using the last 3 terms on the left. We know that  $K_*\underline{bo}_8$  is even degree and  $K_*\underline{BSpin}_1$  is exterior, so the map of the later to the former is trivial. The cokernel is  $K_*\underline{bo}_8$  which is even degree so it injects to  $K_*\underline{bu}_6$ .

5.18.  **$\underline{bo}_9$ .** Consider the fibration

$$K(\mathbb{Z}, 4) \rightarrow \underline{bo}_9 \rightarrow \underline{BSpin}_1.$$

We have evaluated the image of the first map as being the quotient of  $K_*K(\mathbb{Z}, 4)$  by  $K_*K(\mathbb{F}_2, 3)$ , which is just another  $K_*K(\mathbb{Z}, 4)$ . So, the bar spectral sequence  $E^2$  term is just  $K_*\underline{bo}_9$  modulo this image tensored with  $\text{Tor}^{K_*K(\mathbb{F}_2, 3)}$ . However, the answer is exterior and so all of the divided power towers must go away completely. The only way this can happen is if there are differentials,  $d_2$ , from the second filtration generators of these towers to exterior generators in filtration 0. This will allow the exterior generators in Tor to survive. Thus we must have  $K_*\underline{bo}_9$  is  $K_*K(\mathbb{Z}, 4)$  tensor with an exterior algebra. A finite number,  $\binom{n-1}{3}$ , of these exterior generators go to zero in  $K_*\underline{BSpin}_1$  and the same number of exterior generators in  $K_*\underline{BSpin}_1$  are not in the image. [Bicommutativity follows easily from the spectral sequence going from  $K_*\underline{bo}_8$  to  $K_*\underline{bo}_9$ . All the exterior generators are in the image of polynomial generators and so the obstructions are trivial.]

5.19. **Miscellaneous.** The fibration

$$\underline{bo}_4 \rightarrow \underline{bu}_4 \rightarrow \underline{bo}_6$$

gives us a short exact sequence because the homology of  $\underline{bo}_4$  injects.

Considering the fibration

$$BSpin \rightarrow \underline{bu}_4 \rightarrow \underline{BSO}_2,$$

we get a short exact sequence of Hopf algebras. We know all of these are polynomial. If  $K_*BSpin$  injects then we must have split short exact. If it doesn't inject then we know the kernel must be polynomial and so in Tor the higher filtration stuff is generated by exterior generators in the first filtration and they have trivial differentials and the spectral sequence would collapse with odd exterior generators. There are no odd exterior generators in  $K_*\underline{BSO}_2$ , contradiction. Thus there is no kernel.

## 6. BROWN-PETERSON COHOMOLOGY

We would like to have a computation of the Brown-Peterson cohomology of all of the spaces we study in this paper. However, the similar calculation for  $\underline{bu}_k$  for all  $k$  done in [KW01] is extremely difficult and depends on tools we do not have available for the spaces considered here.

Before we proceed we need to review some basic results. The process of going from Morava  $K$ -theory to Brown-Peterson cohomology is developed in [RWY98], [Kas98], [Wil99], [Kas01], and [KW01].

The Brown-Peterson cohomology of many spaces has turned out to be surprisingly nice. The property, in this case, which turns out to qualify as "nice" is Landweber Flatness. A  $BP^*$ -module is said to be *Landweber Flat (LF)* if it is flat for the category of  $BP^*BP$ -modules which are finitely presented over  $BP^*$ . Landweber showed, in [Lan76], that a  $BP^*$ -module,  $M$ , is *LF* if it has no  $p$ -torsion,

$M/(p)M$  has no  $v_1$ -torsion,  $M/(p, v_1)M$  has no  $v_2$ -torsion, etc. This has some very nice properties. In particular, if  $BP^*(X)$  is  $LF$ , then there is a completed Künneth isomorphism for  $BP^*(X \times Y)$ . This makes any such  $X$  which is an  $H$ -space into a completed Hopf algebra as in [KW01].

The first result we need covers most of our spaces. We use the notation  $P(0)$  rather than  $BP$ . It indicates  $BP$  if the inverse limit of the  $BP$  cohomology of the finite skeleta gives the  $BP$  cohomology. This is the case for simple things like  $BO$  and  $BU$ . If not, then we must use the  $p$ -adically complete version of  $BP$ ,  $BP_p^\wedge$ . This happens most of the time for us in this paper, in particular, whenever there is a  $K(\mathbb{Z}, q)$  in a finite Postnikov system.

**Theorem 6.1** ([RWY98]). *If  $K(n)_*(X)$  is concentrated in even degrees for all  $n > 0$  then  $P(0)^*(X)$  is Landweber Flat and it too is concentrated in even degrees.*

All of the even spaces in all four of our Omega spectra have this property so they are all Landweber Flat and are completed Hopf algebras. Also, all of our finite Postnikov systems have this property. In [KW01] it was shown that for the finite Postnikov systems associated with coverings of  $\underline{bu}_i$ , the  $BP$  cohomology split up as the completed tensor product of the  $BP$  cohomology of Eilenberg-Mac Lane spaces having the same homotopy. It is not known if this happens in general (as it does for Morava  $K$ -theory). In particular, we have no such theorem for the finite Postnikov systems which arise in this study.

In [RWY98] there is a basic theorem lifting short exact sequences of Hopf algebras in Morava  $K$ -theory to information about  $BP$  cohomology. This is further refined in [KW01].

**Theorem 6.2** ([RWY98],[KW01, Corollary 6.10, page 79.]). *Let  $E$  be various cohomology theories which includes  $P(0)$ , (see [KW01]), and let the composition of the  $H$ -space maps  $X_1 \rightarrow X_2 \rightarrow X_3$  be trivial. If, for all  $n > 0$ , we have short exact sequences of bicommutative Hopf algebras*

$$K_* \rightarrow K_*X_1 \rightarrow K_*X_2 \rightarrow K_*X_3 \rightarrow K_*$$

*with the middle space having Landweber flat  $E$  cohomology, then the other two spaces also have Landweber flat  $E$  cohomology and we get a similar short exact sequence of completed Hopf algebras in  $E^*(-)$ .*

Theorem 1.9 of the introduction follows immediately from this. Even more. We know from [KW01] that all of the spaces  $\underline{bu}_*$  are Landweber Flat, so the above result combined with Theorem 2.3.6 gives us:

**Theorem 6.3.** *The following fibrations give short exact sequences of completed Hopf algebras for  $BP_2^*(-)$ :  $\underline{bo}_i \rightarrow \underline{bu}_i \rightarrow \underline{bo}_{i+2}$ ,  $\underline{BO}_i \rightarrow \underline{bu}_{i+2} \rightarrow \underline{bo}_{i+2}$ , and  $\underline{BSO}_i \rightarrow \underline{bu}_{i+4} \rightarrow \underline{BO}_{i+2}$  for  $i = 0, 1, 2, 3, 4$  and 8, and  $\underline{BSpin} \rightarrow \underline{bu}_4 \rightarrow \underline{BSO}_2$  and  $\underline{bo}_8 \rightarrow \underline{bu}_6 \rightarrow \underline{BSpin}_2$ .*

Since most of these are spaces with even Morava  $K$ -theory this is straightforward. However, we do get  $\underline{bo}_1$ ,  $\underline{bo}_3$ ,  $\underline{bo}_5$ ,  $\underline{BO}_1$ ,  $\underline{BO}_3$ ,  $\underline{BO}_5$ ,  $\underline{BSO}_1$ , and  $\underline{BSO}_3$  are Landweber Flat without any work. Presumably some of them could be seen to be Landweber Flat by other techniques, but this makes it easy. It is not at all clear how one could get  $\underline{BO}_5$  or  $\underline{BSO}_3$  by other techniques. However, now that we have them, we get more short exact sequences for free. We see that the fibrations  $K(\mathbb{Z}, 4) \rightarrow \underline{BO}_5 \rightarrow \underline{bo}_5$  and  $K(\mathbb{F}_2, 3) \rightarrow \underline{BSO}_3 \rightarrow \underline{BO}_3$  give short exact sequences

of completed Hopf algebras for  $P(0)$  cohomology ( $BP_2^*$  is necessary for the first one but  $BP$  will do for the second).

Our work so far with  $BP$  cohomology has been to collect freebies from our work on Morava  $K$ -theory together with known results. A more thorough understanding of  $BP$  cohomology would require another significant research initiative and this section is not it. We would like to get started on a more explicit study of two of the more important spaces in our paper,  $BSpin$  and  $BO\langle 8 \rangle$ .

*Proof of Theorem 1.11 (ii).* First, we recall Theorem 1.11 (i):

$$BP^*BO \simeq BP^*[[c_1, c_2, \dots]] / (c_i - c_i^*).$$

We have an array of maps with the columns fibrations:

$$\begin{array}{ccccc} BSO & \longrightarrow & BSU & \xrightarrow{1-c} & BSU \\ \downarrow & & \downarrow & & \downarrow \\ BO & \longrightarrow & BU & \xrightarrow{1-c} & BU \\ \downarrow & & \downarrow & & \downarrow \\ RP^\infty & \longrightarrow & CP^\infty & \xrightarrow{1-c} & CP^\infty. \end{array}$$

We know the 3 vertical fibrations give short exact sequences in Morava  $K$ -theory. We know the bottom horizontal fibration gives a short exact sequence. From [RWY98] we know the middle horizontal maps give a left exact sequence. It follows (we are in an abelian category) that the top row is also left exact. All that remains is for us to describe  $c_1(det)$ . Consider

$$S^1 \times \dots \times S^1 \longrightarrow U(n) \xrightarrow{det} S^1$$

where  $det$  is the determinant. The composition from the circles is just the multiplication on the circles. When we take the limit and go to the classifying spaces we can evaluate  $c_1(det)$  as the formal sum  $\sum_i^F x_i$  in  $BP^*(CP^\infty \times \dots \times CP^\infty)$ . This is a symmetric function and so can be written in terms of Conner-Floyd Chern classes. We see that  $BP^*BSU \simeq BP^*BU / (c_1(det))$ .  $\square$

Another way to look at this is as the short exact sequence of completed Hopf algebras coming from the first column:  $BSO \rightarrow BO \rightarrow RP^\infty$ .

$K_*BO$  was computed as the kernel of  $(1-c)_*$  on  $K_*BU$  in [RWY98] by splicing together the two short exact sequences which come from the fibrations  $BO \rightarrow BU \rightarrow \underline{bo}_2$  and  $\underline{bo}_2 \rightarrow BU \rightarrow \underline{bo}_4$  and showing that the composite map was  $1-c$ . We can do the same for  $BSO$  by splicing together the short exact sequences coming from the fibrations  $BSO \rightarrow BSU \rightarrow \underline{BO}_2$  and  $\underline{BO}_2 \rightarrow BSU \rightarrow \underline{bo}_4$ . We can now do this one better: splice together the sequences from the fibrations  $BSpin \rightarrow BSU \rightarrow \underline{BSO}_2$  and  $\underline{BSO}_2 \rightarrow \underline{bu}_6 \rightarrow \underline{BO}_4$ . Together, we get:

**Theorem 6.4.** *Let  $K = K(n)$  ( $p = 2$ ). We have four term exact sequences and maps with the indicated surjections and injections:*

$$\begin{array}{ccccccccc}
 K_* & \longrightarrow & K_* BSpin & \longrightarrow & K_* BSU & \xrightarrow{(1-c)_*} & K_* \underline{bu}_6 & \longrightarrow & K_* BO_4 & \longrightarrow & K_* \\
 & & \downarrow & & \downarrow = & & \downarrow & & \downarrow & & \\
 K_* & \longrightarrow & K_* BSO & \longrightarrow & K_* BSU & \xrightarrow{(1-c)_*} & K_* BSU & \longrightarrow & K_* \underline{bo}_4 & \longrightarrow & K_* \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow = & & \\
 K_* & \longrightarrow & K_* BO & \longrightarrow & K_* BU & \xrightarrow{(1-c)_*} & K_* BU & \longrightarrow & K_* \underline{bo}_4 & \longrightarrow & K_*
 \end{array}$$

*Proof.* The only thing left to do is to justify the name  $1-c$  on the map  $BSU \rightarrow \underline{bu}_6$ . There are fibrations:  $K(\mathbb{Z}, 3) \rightarrow \underline{bu}_6 \rightarrow BSU \rightarrow K(\mathbb{Z}, 4)$ . The map  $1-c$  from  $BSU$  to  $BSU$  pushed down to  $K(\mathbb{Z}, 4)$  is trivial so we get a lift  $BSU \rightarrow \underline{bu}_6$ . Because there are no maps  $BSU \rightarrow K(\mathbb{Z}, 3)$ , this lift is unique.  $\square$

We now have two ways to view  $BP^*BSpin$ . The fibration  $BSpin \rightarrow BSO \rightarrow K(\mathbb{F}_2, 2)$  gives a short exact sequence in Morava  $K$ -theory and so a short exact sequence of completed Hopf algebras when we apply  $BP^*(-)$ . (This proves Theorem 1.11 (iii).) Ideally we would like to evaluate  $BP^*BSpin$  as a quotient of  $BP^*BSO$  in terms of Conner-Floyd Chern classes. A good concrete problem for someone else. We know that  $BP^*K(\mathbb{F}_2, 2)$  has generators in degrees  $2(1 + 2^i)$  with  $i > 0$  from [RWY98, page 191]. One would hope these hit corresponding  $c_k$  with some sort of tail, but precisely what they are we do not know. One might hope to get at this using the other approach that  $BP^*BSpin$  is the cokernel of  $(1-c)^* : BP^*\underline{bu}_6 \rightarrow BP^*BSU$ . Since we know that  $BP^*BSO$  is the cokernel of  $(1-c)^*$  on  $BP^*BSU$ , we are only missing the  $BP^*K(\mathbb{F}_2, 2)$ . We have a short exact sequence from the fibration  $K(\mathbb{Z}, 3) \rightarrow \underline{bu}_6 \rightarrow BSU$  so the  $K(\mathbb{F}_2, 2)$  part must come from the  $K(\mathbb{Z}, 3)$ . In Morava  $K$ -theory we get an injection  $K_*K(\mathbb{F}_2, 2) \rightarrow K_*K(\mathbb{Z}, 3)$  and a corresponding surjection in  $BP_2^*(-)$ . If one lifts elements of  $K_*BSU$  to  $K_*\underline{bu}_6$  and applies  $(1-c)_*$ , one can see the  $K_*K(\mathbb{F}_2, 2)$  part of  $K_*K(\mathbb{Z}, 3)$  get hit. However, it isn't obvious how to use this information to get what we want about  $BP^*BSpin$ .

At this point we know  $BP^*BSO$  explicitly and we know that  $BP^*BSpin$  is  $BP^*BSO // BP^*K(\mathbb{F}_2, 2)$ . We'd like to make progress towards  $BP^*BO\langle 8 \rangle$ . Unfortunately, the technology is not there to use anything more than a short exact sequence to get information. Even at that, the right exact sequence  $K_*BO\langle 8 \rangle \rightarrow K_*BSpin \rightarrow K_*K(\mathbb{F}_2, 3)$  (Theorem 1.5) we have is only algebraic with no geometry behind it. The technology is very far from being able to do anything with this. Consequently we'll have to be a little *ad hoc* in our arguments.

*Proof of Theorem 1.13 (i)-(iii).* We start with a map of fibrations:

$$\begin{array}{ccccc}
 \underline{BSO}_2 & \longleftarrow & \underline{BSpin}_2 & \longleftarrow & K(\mathbb{F}_2, 3) \\
 \uparrow & & \uparrow & & \uparrow \\
 BSU & \longleftarrow & \underline{bu}_6 & \longleftarrow & K(\mathbb{Z}, 3)
 \end{array}$$

The Morava  $K$ -theory of each gives a short exact sequence. Since we are in an abelian category we get a snake lemma, i.e. a 6 term exact sequence of Hopf algebras relating the kernels and the cokernels of the maps between these short exact sequences. As it turns out, we already know all of these kernels and cokernels:

$$\begin{array}{ccccc}
K_* & \longleftarrow & K_* & \longleftarrow & K_*K(\mathbb{F}_2, 3) \\
\uparrow & & \uparrow & & \uparrow = \\
K_*\underline{BSO}_2 & \longleftarrow & K_*\underline{BSpin}_2 & \longleftarrow & K_*K(\mathbb{F}_2, 3) \\
\uparrow & & \uparrow & & \uparrow \rho_* = 0 \\
K_*BSU & \longleftarrow & K_*\underline{bu}_6 & \longleftarrow & K_*K(\mathbb{Z}, 3) \\
\uparrow & & \uparrow & & \uparrow = \\
K_*BSpin & \longleftarrow & K_*BO\langle 8 \rangle & \longleftarrow & K_*K(\mathbb{Z}, 3)
\end{array}$$

The 6 term exact sequence is really a 4 term exact sequence, the one in Theorem 1.5.

Each of these short exact sequences gives a short exact sequence of completed Hopf algebras in  $BP_2^*(-)$ . So, applying  $BP_2^*(-)$  we get:

$$\begin{array}{ccccc}
& & & & BP_2^*K(\mathbb{F}_2, 3) \\
& & & & \downarrow = \\
BP_2^*\underline{BSO}_2 & \longrightarrow & BP_2^*\underline{BSpin}_2 & \longrightarrow & BP_2^*K(\mathbb{F}_2, 3) \\
\downarrow & & \downarrow & & \downarrow 0^* \\
BP_2^*BSU & \longrightarrow & BP_2^*\underline{bu}_6 & \longrightarrow & BP_2^*K(\mathbb{Z}, 3) \\
\downarrow & & \downarrow & \nearrow a & \downarrow 2^* \\
BP_2^*BSpin & \longrightarrow & BP_2^*BO\langle 8 \rangle & \longrightarrow & BP_2^*K(\mathbb{Z}, 3)
\end{array}$$

For clarity the trivial Hopf algebras have been left off of the short exact sequences. The top two rows and the left two columns are all short exact. Some explanation is necessary for what is going on in the lower right hand corner. The map  $2^*$  is not the cokernel. Instead, the bottom row comes from our fibration. We would like to evaluate the map  $BP_2^*BO\langle 8 \rangle \rightarrow BP_2^*K(\mathbb{Z}, 3)$  in the fibration. Because  $BP_2^*\underline{BSpin}_2$  maps trivially through  $BP_2^*\underline{bu}_6$  to  $BP_2^*K(\mathbb{Z}, 3)$ , the map of  $BP_2^*\underline{bu}_6$  to  $BP_2^*K(\mathbb{Z}, 3)$  factors through the cokernel  $BP_2^*BO\langle 8 \rangle$  and we get a surjection  $a$ . We can now do two desirable things. We can mimic parts of both sequences in Theorem 1.5. First, we have the purely algebraic surjection  $BP_2^*BO\langle 8 \rangle \rightarrow BP_2^*K(\mathbb{Z}, 3)$  given by  $a$ . Second, we can splice this together with the short exact sequence

$$BP_2^* \rightarrow BP_2^*K(\mathbb{Z}, 3) \xrightarrow{2^*} BP_2^*K(\mathbb{Z}, 3) \rightarrow BP_2^*K(\mathbb{F}_2, 2) \rightarrow BP_2^*$$

To get a geometrically induced exact sequence of completed Hopf algebras

$$BP_2^*BO\langle 8 \rangle \longrightarrow BP_2^*K(\mathbb{Z}, 3) \longrightarrow BP_2^*K(\mathbb{F}_2, 2) \longrightarrow BP_2^*$$

This has unraveled a significant sticking point. The exact sequences of Theorem 1.5 don't allow us to address the map on  $BP$  to the fibre but we have now managed to see that it is what we would expect it to be anyway.

We can see even more here.  $BP_2^*K(\mathbb{Z}, 3)$  is the cokernel of the map  $BP_2^*BSpin \rightarrow BP_2^*BO\langle 8 \rangle$ . We revert to the basic definition of cokernel to see this. If we have a map (in our category) of  $BP_2^*BO\langle 8 \rangle$  to  $H$  which is trivial on  $BP_2^*BSpin$  then we get a map of  $BP_2^*\underline{bu}_6$  to  $H$  which is trivial on  $BP_2^*BSU$ . Since that sequence is known to be short exact, this means our map factors through  $BP_2^*K(\mathbb{Z}, 3)$ . □

*Proof of Theorem 1.13 (iv)-(vi).* We need to introduce a new space,  $\widetilde{BSpin}$ , the pullback of the maps  $BSpin \rightarrow K(\mathbb{Z}, 4)$  and the Bockstein  $K(\mathbb{F}_2, 3) \rightarrow K(\mathbb{Z}, 4)$ , and a big diagram of fibration sequences:

$$\begin{array}{ccccccc}
 & & & Spin & \longrightarrow & K(\mathbb{Z}, 3) & \\
 & & & \downarrow & & \downarrow 2 & \\
 K(\mathbb{Z}, 2) & \longrightarrow & * & \longrightarrow & K(\mathbb{Z}, 3) & \xrightarrow{=} & K(\mathbb{Z}, 3) \\
 \downarrow \rho & & \downarrow & & \downarrow & & \downarrow \rho \\
 K(\mathbb{F}_2, 2) & \longrightarrow & BO\langle 8 \rangle & \longrightarrow & \widetilde{BSpin} & \longrightarrow & K(\mathbb{F}_2, 3) \\
 \downarrow \beta & & \downarrow = & & \downarrow & & \downarrow \beta \\
 K(\mathbb{Z}, 3) & \longrightarrow & BO\langle 8 \rangle & \longrightarrow & BSpin & \longrightarrow & K(\mathbb{Z}, 4) \\
 \downarrow 2 & & \downarrow & & \downarrow & & \downarrow 2 \\
 K(\mathbb{Z}, 3) & \longrightarrow & * & \longrightarrow & K(\mathbb{Z}, 4) & \xrightarrow{=} & K(\mathbb{Z}, 4)
 \end{array}$$

Our first goal is to show that the two fibrations in boxes give short exact sequences in Morava  $K$ -theory. We show that  $K_*K(\mathbb{Z}, 3)$  injects to  $K_*\widetilde{BSpin}$ . The vertical short exact sequence follows. To get injectivity we use the bar spectral sequence for the fibration  $Spin \rightarrow K(\mathbb{Z}, 3) \rightarrow \widetilde{BSpin}$ . The first map is just the usual map of  $Spin$  to  $K(\mathbb{Z}, 3)$  followed by multiplication by 2. The image of the usual map is  $K_*K(\mathbb{F}_2, 2)$  which is killed by  $2_*$  so the first map is trivial in Morava  $K$ -theory. The cokernel part of the bar spectral sequence is just  $K_*K(\mathbb{Z}, 3)$  which is even degree and therefore injects by Theorem 4.2 (iii).

To see that the (boxed) horizontal fibration gives us a short exact sequence in Morava  $K$ -theory we will again show that the first map is injective on Morava  $K$ -theory. To do this we will use the bar spectral sequence for the fibration  $K(\mathbb{F}_2, 2) \rightarrow BO\langle 8 \rangle \rightarrow \widetilde{BSpin}$ . The first map factors through  $K_*K(\mathbb{Z}, 3)$  and we know that the kernel of this map into  $K_*BO\langle 8 \rangle$  is precisely  $K_*K(\mathbb{F}_2, 2)$  so this map is trivial. Once again we see that the cokernel for the spectral sequence is even degree,  $K_*BO\langle 8 \rangle$ , and so injects.

Since the two boxed fibrations give us short exact sequences in Morava  $K$ -theory, they give short exact sequences of completed Hopf algebras in  $BP_2^*(-)$  as well. We can now evaluate the kernel of the map  $BP_2^*BO\langle 8 \rangle \leftarrow BP_2^*BSpin$ . If we have a map in our category of an  $H$  to  $BP_2^*BSpin$  which is trivial in  $BP_2^*BO\langle 8 \rangle$  then it

maps to  $BP_2^* \widetilde{BSpin}$  and is still trivial in  $BP_2^* BO\langle 8 \rangle$ . Since  $BP_2^* \widetilde{BSpin}$  is part of a short exact sequence we can factor  $H$  through its kernel,  $BP_2^* K(\mathbb{F}_2, 3)$ . This prove part (iv). Parts (v) and (vi) follow from the diagram. We end with our sequence:

$$BP_2^* BSpin \longleftarrow BP_2^* K(\mathbb{Z}, 4) \xleftarrow{2^*} K(\mathbb{Z}, 4) \longleftarrow BP_2^* .$$

□

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