A Warm-up for the Final

1. Prove $\forall n \in \mathbb{N}, \frac{n}{n+1} < 1$.

2. Prove by induction that $\forall n \in \mathbb{N}$ such that $n \geq 3$ an n-element set has $C(n, 2)$ 2-element subsets.

3. Let $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = 3x + 7$ be a function. Prove that $\forall \epsilon > 0, \exists \delta > 0$ such that if $|x - 2| < \delta$, then $|f(x) - f(2)| < \epsilon$.

4. Prove $\forall n \in \mathbb{N}, 3$ divides $2^{2n} - 1$.

5. For all $n \in \mathbb{N}$, define $f_n : \mathbb{R} \to \mathbb{R}$ by $f_n(x) = x^n$. Prove that $\forall n \in \mathbb{N}, \forall x \in \mathbb{R}, \frac{d^n}{dx^n} f_n(x) = n!$.

6. Prove $\forall n \in \mathbb{Z}, 6$ divides $n - n^3$.

7. Prove that $\mathbb{R}$ and $(0, 1)$ have the same cardinality.

8. Suppose that we know the following theorem to be true:

Theorem: Let $H : X \to \mathbb{R}$ be a function. Then the function H has property K, if H is a one-to-one function mapping onto $\mathbb{R}$.

(a) If we let $g : \{x \in \mathbb{R} : x \geq 0\} \to \mathbb{R}$ by $g(x) = x^2 - 4$, then can we conclude from the theorem that
(i) g has property K? Explain.
(ii) g does not have property K? Explain.

(b) Come up with properties K_1 and K_2 so that:
(i) when we replace property K in the above theorem with K_1 it is still true and when we replace K with K_2 it is also true and
(ii) g has property K_1 but not property K_2.