Let $G \subset GL(n, \mathbb{C})$ be Z-closed and $G^* = G$. Set $K = G \cap U(n)$.

Then we know

$$G = K \cdot \underbrace{e^{i\,\mathrm{Lie}(K)}}_{\text{positive definite}}.$$

Let $\langle z, w \rangle = \sum_i z_i \overline{w_i}$. We say that $v \in \mathbb{C}^n$ is <u>critical</u> if

$$\langle Xv, v \rangle = 0, \quad X \in \mathfrak{g} = \mathrm{Lie}(G).$$

Theorem (Kempf - Ness)

(1) $x \in \mathbb{C}^n$ is critical if and only if $\|g \cdot x\| \geq \|x\|$ for all $g \in G$.

(2) If x is critical and $g \in G$ is such that $\|g \cdot x\| = \|x\|$ then $g \cdot x \in K \cdot x$.

(3) $G \cdot x$ contains a critical element if and only if $G \cdot x$ is closed.

$[x, x^*] = 0$.

situation Ex.

Vinberg: If $\mathfrak{g}$ of semi-simple Pair θ cartan relative to $B(x, y^*)$

Then $x \in \mathfrak{g}_{[-1]}$ is critical $\Leftrightarrow$

$f(g \cdot v) = f(v)$

$f\left(\frac{g \cdot v}{\|g \cdot v\|}\right)$

if hom. of degree k

then

$\left|f\left(\frac{g \cdot v}{g \cdot v}\right)\right| = \frac{|f(v)|}{\|g \cdot v\|^k}$

Let $\mathfrak{p} = i\,\mathrm{Lie}(\overset{\mathfrak{k}}{\overset{\shortparallel}{K}}) \subset \mathrm{Lie}(\overset{\mathfrak{g}}{\overset{\shortparallel}{G}})$.

If $X \in \mathfrak{p}$ then $X^* = X$. If $v \in V$ then

$$\frac{d}{dt}\|e^{tX}v\|^2 = \frac{d}{dt}\langle e^{tX}v, e^{tX}v\rangle =$$

$$2\langle Xe^{tX}v, e^{tX}v\rangle \quad (\text{since } X^* = X)$$

$$\frac{d^2}{dt^2}\|e^{tX}v\|^2 = 4\langle Xe^{tX}v, Xe^{tX}v\rangle.$$

If $g \in G$ then $g = ke^X$, $X \in \mathfrak{p}$, $k \in K$ so

$$\|gv\|^2 = \|ke^Xv\|^2 = \|e^Xv\|^2.$$

Let $\alpha(t) = \|e^{tX}v\|^2$. Then

$$\alpha'(t) = 2\langle Xe^{tX}v, e^{tX}v\rangle.$$

If $\|gv\| \geq \|v\|$ all $g \in G$ then $\alpha'(0) = 0 \Rightarrow \langle Xv, v \rangle = 0$ for all X. Since $\mathfrak{g} = \mathfrak{p} + i\mathfrak{p}$, v is critical.

Suppose v is critical. Then $\alpha(t) = \langle k \cdot e^{tX} v, k \cdot e^{tX} v \rangle$ and

$\alpha'(0) = 2\langle Xv, v \rangle = 0$ . .

$\alpha''(0) = 4\langle Xv, Xv \rangle \geq 0$ and 0

$\Longleftrightarrow \|Xv\| = 0$. This implies that

$\|e^{tX} v\| \geq \|v\|$ all $t, X \in \mathfrak{p}$.

$\Longrightarrow \|g \cdot v\| \geq \|v\|$ for all $g \in G$.

We now prove 2. Suppose $\|k e^X v\| = \|v\|$ and v is critical.

Then as usual $\alpha(t) = \langle e^{tX} v, e^{tX} v \rangle$. $\alpha(0) = \alpha(1)$.

But if $Xv \neq 0$ then

$$\alpha''(t) = 4 \langle e^{tX} Xv, e^{tX} Xv \rangle > 0$$

$$\Rightarrow \alpha'(t) > \underbrace{\alpha'(0)}_{=0} \text{ for } t > 0.$$

$$\Rightarrow \alpha(1) > \alpha(0), \Rightarrow\Leftarrow$$

We now prove half of 3.

Suppose that $G \cdot v$ is closed. $(v \neq 0)$ Let $m = \inf_{g \in G} \|g \cdot v\|$.

Then there exist $g_j \in G$ such that $\lim_{j \to \infty} g_j \cdot v = w$ and $\|w\| = m$.

Since $G \cdot v$ closed $w = gv$, $g \in G$.

Thus $m > 0$ and

$$\|g \cdot w\| \geq \|w\|$$

all $g \in G$.

In the case of a Vinberg pair we note that the assertion $x \in V$ critical $\Rightarrow$ $H \cdot x$ is closed is also basically done:

We have observed that in this case we can assume that $\mathfrak{g}$ is semi-simple. Then x critical $\Leftrightarrow$ $[x, x^*] = 0$. Hence $\frac{x+x^*}{2}$ and $\frac{x-x^*}{2i}$ are diagonalizable and commute. We have seen $H \cdot x$ is closed if and only if x is semi-simple.

<u>Example</u>. $V = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$, $G = SL(2,\mathbb{C})^3$.

We note that $v = |000\rangle + |111\rangle$ is critical. Indeed, set $E_{ij}^{(1)} = E_{ij} \otimes I \otimes I$, $E_{ij}^{(2)} = I \otimes E_{ij} \otimes I$, ... then

$$\langle E_{01}^{1} v, v\rangle = \langle v | 011\rangle = 0$$

$$\langle E_{10}^{1} v, v\rangle = \langle v | 100\rangle = 0$$

$$\langle (E_{00}^{1} - E_{11}^{1}) v, v\rangle = \langle v, 000\rangle - \langle v, 111\rangle = 0.$$

$\mathcal{O}(V)^G = \mathbb{C}[f]$, f the triangle $f(v) = 1$. Thus $V/\!/G \cong \mathbb{C}$ so the closed orbits are exactly

$$K = U(2)^3$$

$z \in G \cdot v$, $z \in \mathbb{C}$.

The critical set is $\mathbb{C} \cdot K v$.

Implications consider $|f|_{\text{Sphere}}|$

Then $\left|f\left(\frac{g \cdot v}{\|gv\|}\right)\right| = \frac{1}{\|gv\|^4}$

$\leq \frac{1}{\|v\|^4} \leq \frac{1}{4}$. This implies

$U(2)^3 \cdot \frac{v}{\|v\|} = \left\{ w \in \bigotimes^3 \mathbb{C}^2 \;\middle|\; \|w\| = 1, |f(w)| = \frac{1}{4} \right\}$.

GIT:

Chapter 5

Section 5

$SL(k_1, \mathbb{C}) \otimes \cdots \otimes SL(k_r, \mathbb{C})$

$\mathbb{C}^{k_1} \otimes \cdots \otimes \mathbb{C}^{k_r}$

$|000\ldots0\rangle + |111\ldots\rangle + |k\ldots k\rangle$

k min of $k_1, \ldots, k_r$