let $G \subset GL(n, \mathbb{C})$ be $Z$-closed and $G^* = \mathfrak{g}^*$. Set $K = G \cap U(n)$.

Then we know

$$G = K \cdot \mathbb{C}^n$$

positive definite.

let $\langle z, w \rangle = \sum z_i \overline{w_i}$. We say that $v \in \mathbb{C}^n$ is critical if

$$\langle xv, v \rangle = 0, \quad x \in \mathfrak{g} = \mathfrak{li}(K).$$
Theorem (Kempf - Ness)

1) \( x \in \mathbb{C}^n \) is critical if and only if \( \| g \cdot x \| \geq \| x \| \) for all \( g \in G \).

2) If \( x \) is critical and \( g \in G \) is such that \( \| g \cdot x \| = \| x \| \) then \( g \cdot x \in K \cdot x \).

3) \( G \cdot x \) contains a critical element \( y \) and only if \( G \cdot x \) is closed. \( [x, x^*] = 0 \).

\[ f(\frac{g \cdot v}{\| g \cdot v \|}) = f(v) \]

If hom. of degree \( k \) then

\[ |f(\frac{g \cdot v}{\| g \cdot v \|})| = \frac{|f(v)|}{\| g \cdot v \|} \]
Let $P = \mathfrak{L}(K) \leq \mathfrak{L}(G)$.

If $x \in G$ then $x^* = x$. \hspace{1cm} \forall y \in V \forall h \in V$

\[
\frac{d}{dt} \|e^{tx}v\|^2 = \frac{d}{dt} \langle e^{tx}v, e^{tx}v \rangle = 2 \langle xe^{tx}v, e^{tx}v \rangle \quad \text{(since } x^* = x)\]

\[
\frac{d^2}{dt^2} \|e^{tx}v\|^2 = 4 \langle xe^{tx}v, xe^{tx}v \rangle.
\]

\[
\forall \ g \in G \ \text{then} \ g = ke^x, \ x \in \mathfrak{b}, \ kek \ \text{so}\
\]

\[
\|g\|^2 = \|ke^x\|^2 = \|e^x\|^2.
\]

\[
\langle xe^{tx}v, e^{tx}v \rangle.
\]
If \( \| g \cdot v \| \geq \| v \| \) for all \( g \in G \) then
\[ x'(0) = 0 \Rightarrow \langle x, \nu \rangle = 0. \]
for all \( x \). Since \( g = \theta + i\phi \), \( \nu \)
is critical.
Suppose \( \nu \) is critical. Then
\[ \alpha(t) = \langle k \cdot e^{tx}, k \cdot e^{tx} \rangle \]
and
\[ \alpha'(0) = 2\langle x, \nu \rangle = 0. \]
\[ \alpha''(0) = 4\langle x, x \rangle \geq 0 \text{ and } 0 \]
\( \Leftrightarrow \) \( \| x \| = 0 \). This implies that
\[ \| e^{tx} \cdot v \| \geq \| v \| \text{ all } t, x \in \mathbb{F}. \]
\( \Rightarrow \) \( \| g \cdot v \| \geq \| v \| \) for all \( g \in G \).
We now prove 2. Suppose $\|e^X\| = \|v\|$ and $v$ is critical.

Then as usual $\frac{d}{dt}\alpha(t) = \langle e^{tX}v, e^{tX}v \rangle$. $\alpha(0) = \alpha(1)$.

But if $Xv \neq 0$ then

$$\alpha''(1) = 4 \langle e^{tX}Xv, e^{tX}Xv \rangle > 0$$

$$\Rightarrow \alpha'(1) > \alpha'(0) \text{ Far} \quad \text{for} \quad \delta > 0.$$ 

$$\Rightarrow \alpha(1) > \alpha(0), \quad \Rightarrow$$

We now prove half of 3.
Suppose that $G \cdot v$ is closed. Let $m = \inf_{g \in G} \|g \cdot v\|.$

Then there exist $g_j \in G$ such that

$$\lim_{j \to \infty} g_j \cdot v = w \text{ and } \|w\| = m.$$ 

Since $G \cdot v$ closed, $w = g \cdot v,$ $g \in G.$

Thus $m > 0$ and

$$\|g \cdot w\| \geq \|w\|$$

for all $g \in G.$
In the case of a Virbey pair we note that the assertion \( x \in V \) critical \( \iff \) \( \exists x, x \text{ closed in } V \) also basically done:

We have observed that in this case we can assume that \( G \) is semi-simple. Then \( x \text{ critical } \iff [x, x^*] = 0 \). Hence \( x + x^* \) and \( x - x^* \) are diagonalizable

and commute. We have seen that \( x \text{ is closed if and only if } \)

\( x \text{ is semi-simple.} \)
Example. \( V = \mathbb{C}^2 \otimes \mathbb{C} \otimes \mathbb{C}^2 \), \( G = \text{SL}(3, \mathbb{C}) \).

We note that \( V = |10000\rangle + |1111\rangle \) is critical. Indeed, set \( E_{ij}^{(1)} = E_{i}^1 \otimes I \otimes I \)
\( E_{ij}^{(2)} = I \otimes E_j^2 \otimes I \), ... Then
\[
\langle E_0^{(1)} v, v \rangle = \langle v | 1011 \rangle = 0
\]
\[
\langle E_1^{(1)} v, v \rangle = \langle v | 1100 \rangle = 0
\]
\[
\langle (E_0^{(2)} - E_1^{(2)}) v, v \rangle = \langle v | 000 \rangle - \langle v | 111 \rangle = 0.
\]

\( \Theta(V)^G = C[f] \), \( f \) the triangle
\( f(v) = 1 \). Thus \( V//G = C \) so
The irreducible orbits are exactly
$K = \mathcal{U}(n^2)$

$z \in G \cdot v, \quad z \in \mathbb{C}$.

The critical set is $C \cdot v$.

Implications consider $H \mid Spherical$.

Then $|f(g \cdot v)| = \frac{1}{|v|} |g|^4$

$$\leq \frac{1}{|v|^4} \leq \frac{1}{4}.$$ This implies

$U(2)_v \cdot \frac{v}{|v|^4} = \{ w \in \mathbb{C}^2 \mid |w| = 1, |f(w)| = \frac{1}{4} \}$. 

$SL(k, \mathbb{C}) \otimes \cdots \otimes SL(k, \mathbb{C})$

Chapter 5

Section 5