If W is a vector space over $\mathbb{C}$ of dimension n. Then a transformation $\sigma: W \to W$ is called a complex reflection if $\dim \ker(\sigma - I) = n-1$ and $\sigma^m = 1$ for some $0 < m < \infty$.

So there is a basis such that

$$\sigma \sim \begin{bmatrix} \zeta & 0 & 0 & \cdots & 0 \\ & 1 & & & 0 \\ & & 1 & & \\ 0 & & & \ddots & \\ & & & & 1 \end{bmatrix}$$

$\zeta^m = 1$.

The main theorem of Vinberg is that the Weyl group of a Vinberg pair is generated by complex reflections.

We will assume that our Vinberg pair is as in the last lecture. The key step is

Theorem. $(\mathfrak{a}-\{0\})/W_{\mathbb{H}}(\mathfrak{a})$ is simply connected if $\dim \mathfrak{a} > 1$.

① $(Crit(V)-\{0\})/K_{\mathbb{H}}$ is homeomorphic with $(V-\{Nullcone\})/\!/_{\mathbb{H}}$ which is is analytic with $(\mathcal{O}_Z-\{0\})/W_{\mathbb{H}}(\alpha)$.

This is proved in the GIT MS in the last section of chapter 3.

② $(Crit(V)-\{0\})/K_{\mathbb{H}}$ is a deformation retract of $(V-\{Nullcone\})/K_{\mathbb{H}}$

We show how this proves the theorem then we will sketch its proof.

① implies that $(\mathrm{crit}(V)-\{0\})/K_H$
is homeomorphic with $(\mathbb{R}-\{0\})/W$.
② implies
Thus $(V-\text{Nullcone})/K_H$ is homeomorphic
with $(\mathbb{R}-\{0\})/W$.

Now K_H is a connected compact Lie group
and Montgomery-Zippin have
shown that if Z is a simply connected
path connected space and U is a
compact connected
Lie group acting on Z then
if Z is simply connected so is Z/U.

Thus we must see that
V - NullCone is simply connected.
But the null cone has real codimension
$\dim_{\mathbb{R}} \geq 4$ so since V is simply
connected so is V - NullCone.

We are left with 2. We recall
that $\mathrm{Crit}(V) = \{x \in V \mid [x, x^*] = 0\}$.
We consider the following system
of ODE on V:

$$\frac{dF}{dt} = -[[F, F^*], F].$$

We note that if $X \in M_n(\mathbb{C})$ and if $u \in U(n)$ then $[[uXu^{-1}, (uXu^{-1})^*], uXu^{-1}]$

$(uXu^{-1})^* = uX^*u^{-1}$

$= u[[X, X^*], X]u^{-1}$.

Thus if $k \in k_{+1} = f^{-1} \cap U(n)$ we have

$$k \cdot \frac{dF}{dt} = -[[k \cdot F, (k \cdot F)^*], k \cdot F]$$

Thus if we denote the solution near $t = 0$ with

initial condition $F(0) = V$ by
$F_t(v)$ then $F_t(k \cdot v) = k \cdot F_t(v)$
$k \in K_{+1}$.

We calculate

$$\frac{d}{dt} \langle F_t(v), F_t(v) \rangle =$$

$$- \langle \left[\left[\bar{F}_t(v), F_t(v)^* \right], F_t(v) \right], F_t(v) \rangle$$

$$= -\mathrm{tr}\Big(\qquad\qquad F_t(v)^* \Big) - \Big(\qquad\qquad \Big)$$

$$= -2\mathrm{tr}\ [F_t(v), F_t(v)^*], [F_t(v), F_t(v)^*] \leq 0$$

$$\Longrightarrow \|F_t(v)\| \leq \|v\| \quad \forall$$

$t \geq 0$.

Thus if $B_r = \{ v \in V \mid \|v\| \leq r \}$

$F_t(v) \subset B_r$ where defined for $t \geq 0$.

This implies that $F_t(v)$ is

defined for $t \geq 0$.

Observing that if $R(v) = -[v, v^*]$

then $\frac{d}{dt} F_t = [R(F_t), F_t]$

and $R(v) \in \mathrm{lie}(H)$. So

$F_t(v) \in$ H.v. for all $t \geq 0$.

For the moment fix v
and set $F(t) = F_t(v)$. Then

$$H(t) = \langle [F(t), F(t)^*], [F(t), F(t)^*] \rangle$$

$$\text{Then } \frac{dH}{dt} = -\langle [[[F, F^*], F], F^*], [F(t), F(t)^*] \rangle \quad (A)$$

$$- \langle [F, [[F, F^*], F]^*], [F, F^*] \rangle \quad (B)$$

$$- \overline{A} - \overline{B}$$

$$= -4 \| [[F, F^*], F] \|^2.$$

$$= -4 \| \frac{dF}{dt} \|^2.$$

We also note that

$$h(x) = \langle [x,x^*],[x,x^*]\rangle =$$

$$\operatorname{tr}\left([x,x^*]\cdot[x,x^*]\right) =$$

$$\operatorname{tr}\left([[x,x^*],x]\cdot x^*\right) = \langle [[x,x^*],x],x^*\rangle$$

So $$|H(t)| \le \frac{1}{2}|H'(t)|^{\frac{1}{2}}\|F(t)\| \le \cdots$$

$$\frac{1}{2}\|v\|\,|H'(t)|^{1/2}$$

so if $V \neq 0$

$$|H'(t)|^{1/2} \ge \frac{2}{\|v\|}H(t)$$

$$\Rightarrow -H'(t) \ge \frac{4}{\|v\|^2}H(t)^2$$

Now

$$\frac{d}{dt} H(t)^{-1} = -\frac{H'(t)}{H(t)^2} \geq \frac{4}{\|v\|^2}$$

Thus $H(t)^{-1} \geq \frac{4}{\|v\|^2} t$. Thus

$$H(t) \leq \frac{\|v\|^2}{4} t^{-1}.$$

This implies that

$$\lim_{t \to +\infty} \| [F_t(v), F_t(v)^*] \| = 0$$

uniformly on B_r for each $r > 0$.

Noting that $\|F_t(v)\| \le \|v\|$

and that $F_t(v) \in \overline{H\cdot v}$ all $t \ge 0$

we have if $F_{t_j}(v)$ is a convergent

subsequence ~~then~~ ($t_j \to \infty$)

$$z = \lim_{j\to\infty} F_{t_j}(v) \in \overline{H\cdot v}$$

and $\sigma = \lim_{j\to\infty} [F_{t_j}(v), F_{t_j}(v)^*] \longrightarrow [z, z^*]$.

Thus $z \in Cent(V) \cap \overline{H\cdot v} = K_H \cdot z$

with $\|z\| = \min_{w \in \overline{H\cdot v}} \|w\|$.

Thus $\lim_{t\to+\infty} K_H \cdot F_t(v) = \{$unique K_H-orbit in $Cent(V) \cap \overline{H\cdot v}\}$.

uniformly on B_r.

QED.

$$\frac{dX}{dt} = -[[X, X^*], X] \qquad M_n(\mathbb{C})$$

$$\lim_{t \to \infty} \psi_t(v) = \psi_\infty(v) \quad \underline{\text{normal}}$$

$$\{ u \, \psi_\infty(v) \, u^{-1} \mid u \in U(n) \}$$

$=$ The set of normal matrices

in $\{ g v g^{-1} \mid g \in GL(n, \mathbb{C}) \}$.