$V$ be a vector space over $\mathbb{C}$ of dimension $n < \infty$.

$\mathcal{O}(V)$ denote the algebra of all polynomial functions on $V$.

If $G \subset GL(V)$ is a subgroup then we set $\mathcal{O}(V)^G$ equal to the subalgebra of polynomials such that if $g \in G, v \in V$ then $gf(v) := f(g^{-1}v) = f(v)$.

Suppose that $W$ is a finite subgroup of $GL(V)$.

**Theorem** If $f(x) = f(y)$ for all $f \in \mathcal{O}(V)^W$ then $y \in Wx$. 
Exercises 1. If $A$ and $B$ are finite, disjoint subsets of $V$ then there exists $f \in \mathcal{O}(V)$ such that $f(a) = 1, a \in A$ and $f(b) = 0, b \in B$.

2. Set $A = Wx$ and $B = Wy$ and assume $A \cap B = \emptyset$. Let $f$ be as in the previous exercise. If $h(z) = \frac{1}{|W|} \sum_{w \in W} f(wz)$. Then $h \in \mathcal{O}(V)^W$ and $h|_A = 1$, $h|_B = 0$. 
**Theorem.** \( \mathcal{O}(V)^W \) has a finite set of homogeneous generators.

**Exercise.** Let \( \mathcal{O}(V)^W_+ = \{ f \in \mathcal{O}(V)^W | f(0) = 0 \} \). The ideal \( I = \mathcal{O}(V)\mathcal{O}(V)^W_+ \) has homogeneous generators \( u_1, ..., u_d \in \mathcal{O}(V)^W_+ \). Show that \( \{ u_1, ..., u_d \} \) generates \( \mathcal{O}(V)^W \).

We note that the algebra \( \mathcal{O}(V) \) is integral over \( \mathcal{O}(V)^W \).

**Exercise.** If \( f \in \mathcal{O}(V) \) define \( p(t) = \prod_{s \in W} (t - sf) \) show that \( p(t) = t^{|W|} + \sum_{j=0}^{|W|-1} t^j a_j \) with \( a_j \in \mathcal{O}(V)^W \). Observe that \( p(f) = 0 \).

This implies that if \( x_1, ..., x_n \) are coordinates on \( V \) then there exist relations

\[
x_i^{k_i} = \sum_{j=0}^{k_i-1} a_j x_i^j.
\]
with $a^i_j \in \mathcal{O}(V)^W$. Hence

$$\mathcal{O}(V) = \mathcal{O}(V)^W + \sum_{i=1}^n \sum_{j=1}^{k_i-1} \mathcal{O}(V)^W x_{i,j}.$$

Nakayama’s Lemma implies that if $\mathfrak{m}$ is a maximal ideal in $\mathcal{O}(V)^W$ then there exists $\mathcal{M}$ a maximal ideal in $\mathcal{O}(V)$ (that is $\mathfrak{m}_a = \{ f \in \mathcal{O}(V) | f(a) = 0 \}$ by the Nullstellensatz) such that $\mathcal{M} \cap \mathcal{O}(V)^W = \mathfrak{m}$. 
Nakayama’s Lemma: $B \supset A$ rings with 1 such that $B = \sum_{i=1}^{m} Ab_i$. If $I$ is a proper ideal in $A$ then $IB \neq B$.

**Proof.** If $IB = B$ then we show that $1 \in T$.

$$b_i = \sum_j a_{ij}b_j$$

with $a_{ij} \in I$. Thus

$$\begin{bmatrix} a_{ij} - \delta_{ij} \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} = 0.$$ Multiplting by the classical adjoint of $\begin{bmatrix} a_{ij} - \delta_{ij} \end{bmatrix}$ we have by Cramer’s rule

$$\det \begin{bmatrix} a_{ij} - \delta_{ij} \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} = 0.$$ Thus $\begin{bmatrix} a_{ij} - \delta_{ij} \end{bmatrix} = 0$. Expanding the determinant this shows $1 \in I$. ■
Let $f_1, \ldots, f_m$ be a set of homogeneous generators of $\mathcal{O}(V)^W$. We set $F(v) = (f_1(v), \ldots, f_m(v))$. Then $F$ pushes down to

$$F : V/W \to \mathbb{C}^m$$

and on $V/W$ it is injective. Let

$$\mathcal{I} = \{ \phi \in \mathcal{O}(\mathbb{C}^m) | \phi(F(V)) = 0 \}.$$

If $X = \{ x \in \mathbb{C}^m | \mathcal{I}(x) = 0 \}$ and if $\mathcal{O}(X) = \mathcal{O}(\mathbb{C}^m)|_X$ then

$$F^* \mathcal{O}(X) = \mathcal{O}(V)^W.$$

$$F^* \phi = \phi \circ F.$$

Let $x \in X$ then the ideal in $\mathcal{O}(X)$, $m_x = \{ f \in \mathcal{O}(X) | f(x) = 0 \}$, is a maximal ideal. Now $F^* m_x$ is a maximal ideal in $\mathcal{O}(V)^W$. Thus there exists $a \in V$ such that $F^* m_x = m_a \cap \mathcal{O}(V)^W$. We have $F(a) = x$.

So $F(V/W) = X$. This gives $V/W$ the structure of an affine variety.
A subset $X$ of $V$ is said to be Zariski closed if it is the locus of zeros of a set of polynomials on $V$.

**Exercise.** This collection of closed sets satisfies the definition of closed sets for a topology. This is $\emptyset$ and $V$ are closed, a finite union of closed sets is closed and an arbitrary intersection of closed sets is closed.

If $X$ is Zariski closed we endow $X$ with the subspace topology. If $X \subset V$ is closed then we set

$$ \mathcal{I}(X) = \{ f \in \mathcal{O}(V) | f|_X = 0 \}. $$

As an algebra $\mathcal{O}(X) := \mathcal{O}(V)|_X \cong \mathcal{O}(V)/\mathcal{I}(X)$. A pair of a set, $Y$, an algebra of functions $A$, on $Y$ is called an affine variety if there exists a finite dimensional vector space $V$, a Zariski closed subset $X$ in $V$ and a bijective map $\Phi : Y \to X$ such that $\Phi^* : \mathcal{O}(X) \to A$ is an algebra homomorphism.
An element, $\sigma$, of $GL(V)$ is called a complex reflection if there exists $m > 0$ such that $\sigma^m = I$ and

$$\dim \ker(\sigma - I) = n - 1.$$ 

If $m = 2$ then $\sigma$ is called a reflection.

A complex reflection group is a finite subgroup of $GL(V)$ generated by complex reflections.

**Example.** $W$ the group of transformations $\sigma(x_1, \ldots, x_n) = (x_{\sigma^{-1}1}, \ldots, x_{\sigma^{-1}n})$ acting on $\mathbb{C}^n$. If $\sigma$ interchanges exactly 2 coordinates, $i \neq j$, then $\sigma^2 = I$ and $\ker(\sigma - I) = \{(x_1, \ldots, x_n)|x_j = x_i\}$. $W$ is generated by these reflections.
Example. \( \zeta = e^{\frac{2\pi i}{3}} = \frac{-1+\sqrt{3}}{2} \).

\[
\begin{align*}
  s_1 &= \begin{bmatrix} 1 & 1 \\ 1 & \zeta^2 \end{bmatrix}, \\
  s_2 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \zeta^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\
  s_3 &= \frac{-i\zeta}{\sqrt{3}} \begin{bmatrix} 1 & \zeta & \zeta \\ \zeta & 1 & \zeta \\ \zeta & \zeta & 1 \end{bmatrix}
\end{align*}
\]

Show that \( s_3 \) is a reflection. The group generated by these reflections is of order 648 (we will see why later).
The Shephard-Todd theorem says

**Theorem**

1. If $W \subset GL(V)$ is a complex reflection group then $\mathcal{O}(V)^W$ is generated by $n$ algebraically independent homogeneous polynomials.

2. If $\mathcal{O}(V)^W$ is generated by $n$ algebraically independent homogeneous polynomials then $W \subset GL(V)$ is a complex reflection group.

**Example.** If $W$ is as above taking for $k = 1, \ldots, n$

\[ e_k(x) = \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq n} x_{i_1} \cdots x_{i_k} \]

(i.e. $\prod_{i=1}^{n}(1 + tx_i) = \sum_{k=0}^{n} t^k e_k, e_0 = 1$) form an algebraically independent set of generators.
Exercise 1. Prove this directly. Here are some suggestions (you should prove any unproved statements). Order the monomials \( x^I = x_1^{i_1} \cdots x_n^{i_n} \) using graded lexicographic order. So \( x^I > x^J \) if degree \( x^I = |I| = i_1 + \cdots + i_m > |J| \) and if \( |I| = |J| \) then \( i_1 = j_1, \ldots, i_r = j_r \) and \( i_{r+1} > j_{r+1} \). If \( f \) is a polynomial in \( x_1, \ldots, x_n \) then

\[
f(x) = a_I x^I + \sum_{J < I} a_J x^J \tag{*}
\]

with \( a_I \neq 0 \). We call \( I \) the leading exponent of \( f \). If \( f \) is invariant under the action of \( S_n \) then the leading exponent of \( f \) is dominant, that is \( i_1 \geq i_2 \geq \cdots \geq i_n \). If \( I \) is dominant then

\[
\sigma_I = e_1^{i_1-i_2} e_2^{i_2-i_3} \cdots e_n^{i_n}
\]

has leading exponent \( I \). We prove the result assuming the result is false and deriving a contradiction. Let \( f \) be a \( S_n \) polynomial with minimal leading exponent \( I \) such that \( f \) can’t be written as a polynomial in \( e_1, \ldots, e_n \). If \( f \) is written as in (\( \ast \)) then \( f - a_I \sigma_I \) has leading exponent strictly less than \( f \). Which implies the desired contradiction.
Exercise 2. Show that \( x_1^6 + x_2^6 + x_3^6 - 10(x_1^3 x_2^3 + x_2^3 x_3^3 + x_1^3 x_3^3) \) is an invariant polynomial for the group defined by the third order reflections above (number 25 on Shephard-Todd's list).
Shephard and Todd proved their theorem in three steps. First they classified the complex reflection groups. Step 2 proved 1. for each one. Step 3 was to prove 2. using 1. This step didn’t explicitly use the classification. We will give a variant of their proof of 1. implies 2. involving some ideas of Springer which also proves:

**Theorem** Let $W$ be a finite subgroup of $GL(n, \mathbb{C})$ and let $g_1, \ldots, g_n$ be algebraically independent homogeneous elements of $O(\mathbb{C}^n)^W$. If $d_i = \deg g_i$ then $|W| \leq d_1 \cdots d_n$. Equality implies that $W$ is generated by reflections and $O(\mathbb{C}^n)^W = \mathbb{C}[g_1, \ldots, g_n]$.

**Example.** $W = S_n$ acting as above. $g_1, \ldots, g_n$ the elementary symmetric functions. They are algebraically independent and $\prod_{i=1}^n \deg g_i = n!$. So $O(\mathbb{C}^n)^W = \mathbb{C}[g_1, \ldots, g_n]$.

Chevalley, gave a proof of 1. for finite groups generated by reflections. Serre (and others) pointed out that his proof with essentially no change worked to prove I. in full generality.