V be a vector space over $\mathbb{C}$ of dimension $n < \infty$.

$\mathcal{O}(V)$ denote the algebra of all polynomial functions on V.

If $G \subset GL(V)$ is a subgroup then we set $\mathcal{O}(V)^G$ equal to the subalgebra of polynomials such that if $g \in G, v \in V$ then $gf(v) := f(g^{-1}v) = f(v)$.

Suppose that W is a finite subgroup of $GL(V)$.

Theorem If $f(x) = f(y)$ for all $f \in \mathcal{O}(V)^W$ then $y \in Wx$.

Exercises 1. If A and B are finite, disjoint subsets of V then there exists $f \in \mathcal{O}(V)$ such that $f(a) = 1, a \in A$ and $f(b) = 0, b \in B$.

2. Set $A = Wx$ and $B = Wy$ and assume $A \cap B = \emptyset$. Let f be as in the previous exercise. If $h(z) = \frac{1}{|W|} \sum_{w \in W} f(wz)$. Then $h \in \mathcal{O}(V)^W$ and $h_{|A} = 1$, $h_{|B} = 0$.

Theorem $\mathcal{O}(V)^W$ has a finite set of homogeneous generators.

Exercise. Let $\mathcal{O}(V)^W_+ = \{f \in \mathcal{O}(V)^W | f(0) = 0\}$. The ideal $I = \mathcal{O}(V)\mathcal{O}(V)^W_+$ has homogeneous generators $u_1, ..., u_d \in \mathcal{O}(V)^W_+$. Show that $\{u_1, ..., u_d\}$ generates $\mathcal{O}(V)^W$.

We note that the algebra $\mathcal{O}(V)$ is integral over $\mathcal{O}(V)^W$.

Exercise. If $f \in \mathcal{O}(V)$ define $p(t) = \prod_{s \in W}(t - sf)$ show that $p(t) = t^{|W|} + \sum_{j=0}^{|W|-1} t^j a_j$ with $a_j \in \mathcal{O}(V)^W$. Observe that $p(f) = 0$.

This implies that if $x_1, ..., x_n$ are coordinates on V then there there exist relations

$$x_i^{k_i} = \sum_{j=0}^{k_i-1} a_j^i x_i^j.$$

with $a_j^i \in \mathcal{O}(V)^W$. Hence

$$\mathcal{O}(V) = \mathcal{O}(V)^W + \sum_{i=1}^{n} \sum_{j=1}^{k_i-1} \mathcal{O}(V)^W x_i^j.$$

Nakayama's Lemma implies that if $\mathfrak{m}$ is a maximal ideal in $\mathcal{O}(V)^W$ then there exists $\mathcal{M}$ a maximal ideal in $\mathcal{O}(V)$ (that is $\mathfrak{m}_a = \{f \in \mathcal{O}(V) | f(a) = 0\}$ by the Nullstellensatz) such that $\mathcal{M} \cap \mathcal{O}(V)^W = \mathfrak{m}$.

Nakayama's Lemma: $B \supset A$ rings with 1 such that $B = \sum_{i=1}^{m} Ab_i$. If I is a proper ideal in A then $IB \neq B$.

Proof. If $IB = B$ then we show that $1 \in T$.

$$b_i = \sum_j a_{ij} b_j$$

with $a_{ij} \in I$. Thus

$$\left[a_{ij} - \delta_{ij}\right] \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} = 0.$$

Multiplting by the classical adjoint of $\left[a_{ij} - \delta_{ij}\right]$ we have by Cramer's rule

$$\det \left[a_{ij} - \delta_{ij}\right] \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} = 0.$$

Thus $\left[a_{ij} - \delta_{ij}\right] = 0$. Expanding the determinant this shows $1 \in I$. ■

Let $f_1, ..., f_m$ be a set of homogeneous generators of $\mathcal{O}(V)^W$. We set $F(v) = (f_1(v), ..., f_m(v))$. Then F pushes down to

$$F : V/W \to \mathbb{C}^m$$

and on V/W it is injective. Let

$$\mathcal{I} = \{\phi \in \mathcal{O}(\mathbb{C}^m) | \phi(F(V)) = 0\}.$$

If $X = \{x \in \mathbb{C}^m | \mathcal{I}(x) = 0\}$ and if $\mathcal{O}(X) = \mathcal{O}(\mathbb{C}^m)_{|X}$ then

$$F^*\mathcal{O}(X) = \mathcal{O}(V)^W.$$

$$F^*\phi = \phi \circ F.$$

Let $x \in X$ then the ideal in $\mathcal{O}(X)$, $\mathfrak{m}_x = \{f \in \mathcal{O}(X) | f(x) = 0\}$, is a maximal ideal. Now $F^*\mathfrak{m}_x$ is a maximal ideal in $\mathcal{O}(V)^W$. Thus there exists $a \in V$ such that $F^*\mathfrak{m}_x = \mathfrak{m}_a \cap \mathcal{O}(V)^W$. We have $F(a) = x$.

So $F(V/W) = X$.This gives V/W the structure of an affine variety.

A subset X of V is said to be Zariski closed if it is the locus of zeros of a set of polynomials on V.

Exercise. This collection of closed sets satisfies the definition of closed sets for a toplolgy. This is $\emptyset$ and V are closed, a finite union of closed sets is closed and an arbitrary intersection of closed sets is closed.

If X is Zariski closed we endow X with the subspace topology. If $X \subset V$ is closed then we set

$$\mathcal{I}(X) = \{f \in \mathcal{O}(V) | f_{|X} = 0\}.$$

As an algebra $\mathcal{O}(X) := \mathcal{O}(V)_{|X} \cong \mathcal{O}(V)/\mathcal{I}(X)$. A pair of a set, Y, an an algebra of functions A,on Y is called an affine variety if there exists a finite dimensional vector space V, a Zariski closed subset X in V and a bijective map $\Phi : Y \to X$ such that $\Phi^* : \mathcal{O}(X) \to A$ is an algebra homorphism.

An element, σ, of $GL(V)$ is called a complex reflection if there exists $m > 0$ such that $\sigma^m = I$ and

$$\dim \ker(\sigma - I) = n - 1.$$

If $m = 2$ then σ is called a reflection.

A complex reflection group is a finite subgroup of $GL(V)$ generated by complex reflections.

Example. W the group of transformations $\sigma(x_1, ..., x_n) = (x_{\sigma^{-1}1}, ..., x_{\sigma^{-1}n})$ acting on $\mathbb{C}^n$. If σ interchnges exactly 2 coordinates, $i \neq j$, then $\sigma^2 = I$ and $\ker(\sigma - I) = \{(x_1, ..., x_n) | x_j = x_i\}$. W is generated by these reflections.

Example. $\zeta = e^{\frac{2\pi i}{3}} = \frac{-1+\sqrt{3}}{2}$.

$$s_1 = \begin{bmatrix} 1 & & \\ & 1 & \\ & & \zeta^2 \end{bmatrix},$$

$$s_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \zeta^2 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$s_3 = \frac{-i\zeta}{\sqrt{3}} \begin{bmatrix} 1 & \zeta & \zeta \\ \zeta & 1 & \zeta \\ \zeta & \zeta & 1 \end{bmatrix}$$

Show that s_3 is a reflection. The group generated by these reflections is of order 648 (we will see why later).

The Shephard-Todd theorem says

Theorem

1. If $W \subset GL(V)$ is a complex reflection group then $\mathcal{O}(V)^W$ is generated by n algebraically independent homogeneous polynomials.

2. If $\mathcal{O}(V)^W$ is generated by n algebraically independent homogeneous polynomials then $W \subset GL(V)$ is a complex reflection group.

Example. If W is as above taking for $k = 1, ..., n$

$$e_k(x) = \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq n} x_{i_1} \cdots x_{i_k}$$

(i.e. $\prod_{i=1}^{n}(1 + tx_i) = \sum_{k=0}^{n} t^k e_k, e_0 = 1$) form an algebraically independent set of generators.

Exercise 1. Prove this directly. Here are some suggestions (you should prove any unproved statements). Order the monomials $x^I = x_1^{i_1} \cdots x_n^{i_n}$ using graded lexicographic order. So $x^I > x^J$ if degree $x^I = |I| = i_1 + ... + i_m > |J|$ and if $|I| = |J|$ then $i_1 = j_1, ..., i_r = j_r$ and $i_{r+1} > j_{r+1}$. If f is a polynomial in $x_1, ..., x_n$ then

$$f(x) = a_I x^I + \sum_{J<I} a_J x^J (*)$$

with $a_I \neq 0$. We call I the leading exponent of f. If f is invariant under the action of S_n then the leading exponient of f is dominant, that is $i_1 \geq i_2 \geq ... \geq i_n$. If I is dominant then

$$\sigma_I = e_1^{i_1 - i_2} e_2^{i_2 - i_3} \cdots e_n^{i_n}$$

has leading exponent I. We prove the result assuming the result is false and deriving a contrdiction. Let f be a S_n polynomial with minimal leading exponent I such that f can't be written as a polynomial in $e_1, ..., e_n$. If f is written as in $(*)$ then $f - a_I \sigma_I$ has leading exponent strictly less than f. Which implies the desired contradiction.

Exercise 2. Show that $x_1^6 + x_2^6 + x_3^6 - 10(x_1^3x_2^3 + x_2^3x_3^3 + x_1^3x_3^3)$ is an invariant polynomial for the group defined by the third order reflections above (number 25 on Shephard-Todd's list).

Shephard and Todd proved their theorem in three steps. First they classified the complex reflection groups. Step 2 proved I. for each one. Step 3 was to prove 2. using 1. This step didn't explicitly use the classification. We will give a variant of their proof of 1. implies 2. involving some ideas of Springer which also proves:

Theorem Let W be a finite subgroup of $GL(n, \mathbb{C})$ and let $g_1, ..., g_n$ be algebraically independent homogeneous elements of $\mathcal{O}(\mathbb{C}^n)^W$. If $d_i = \deg g_i$ then $|W| \leq d_1 \cdots d_n$. Equality implies that W is generated by reflections and $\mathcal{O}(\mathbb{C}^n)^W = \mathbb{C}[g_1, ..., g_n]$.

Example. $W = S_n$ acting as above. $g_1, ..., g_n$ the elementary symmetric functions. They are algebraically independent and $\prod_{i=1}^n \deg g_i = n!$. So $\mathcal{O}(\mathbb{C}^n)^W = \mathbb{C}[g_1, ..., g_n]$.

Chevalley, gave a proof of 1. for finite groups generated by reflections. Serre (and others) pointed out that his proof with essentially no change worked to prove I. in full generality.