Solutions for Midterm (Math 109)

November 8, 2007

1.

P	Q	not (P and Q)	P $\Longrightarrow$ (not Q)	(not P) or (not Q)	(not Q) $\Longrightarrow$ (not P)
T	T	F	F	F	T
T	F	T	T	T	F
F	T	T	T	T	T
F	F	T	T	T	T

The contrapositive of P $\Longrightarrow$ Q is (not Q) $\Longrightarrow$ (not P).

2. **Claim**: If A and B are finite sets with $B \subseteq A$ then $|A - B| = |A| - |B|$.

Proof: First suppose $A - B = \emptyset$. Let $a \in A$. Since $A - B = \emptyset$ we have that $a \in B$. Therefore $A \subseteq B$. Combining this with our hypothesis that $B \subseteq A$ we get that $A = B$. Then in particular $|A| = |B|$, or in other words $|A| - |B| = 0$. And by assumption $|A - B| = 0$ since the cardinality of the empty set is zero. Therefore $|A - B| = |A| - |B|$.

Now suppose $A - B \neq \emptyset$. First note that if $B = \emptyset$ then the claim is trivial. So suppose also that $B \neq \emptyset$. Let $A - B$ have cardinality n and B have cardinality m, where n and m are natural numbers. Then there exist bijections

$$\begin{aligned} f &: \mathbb{N}_n \longrightarrow A - B \\ g &: \mathbb{N}_m \longrightarrow B \end{aligned}$$

We want to show that the cardinality of A is $n + m$, and to do that we have to construct a bijection

$$h : \mathbb{N}_{n+m} \longrightarrow A$$

To define h we have to use f and g:

$$h(j) = \begin{cases} f(j) & \text{if } 1 \leq j \leq n \\ g(j - n) & \text{if } n + 1 \leq j \leq n + m \end{cases}$$

To see that h is onto note that if $a \in A$ then either $a \in A - B$ or $a \in B$. If $a \in A - B$ then there exists $j \in \mathbb{N}_n$ such that $f(j) = a$. Then $h(j) = f(j) = a$.

If $a \in B$ then there is $k \in \mathbb{N}_m$ such that $g(k) = a$. Then $h(k+n) = g(k) = a$. So indeed h is onto.

To see that h is one-to-one let $i, j \in \mathbb{N}_{n+m}$ be distinct. We have to show that $h(i) \neq h(j)$. This breaks into three cases. First suppose $i, j \in \mathbb{N}_n$. Then $h(i) = f(i)$ and $h(j) = f(j)$. Since f is one-to-one $f(i) \neq f(j)$. Therefore $h(i) \neq h(j)$. Secondly suppose $i, j > n$. Then $h(i) = g(i-n)$ and $h(j) = g(j-n)$. Since g is one-to-one $g(i-n) \neq g(j-n)$. Therefore $h(i) \neq h(j)$. Finally suppose $i \leq n$ and $j > n$. Then $h(i) = f(i)$ and $h(j) = g(j-n)$. Since $f(i) \in A - B$ and $g(j-n) \in B$ and $(A-B) \cap B = \emptyset$ we must have that $f(i) \neq g(j-n)$. Therefore $h(i) \neq h(j)$. So indeed h is one-to-one.

The three bijections f, g, and h show respectively that $|A-B| = n$, $|B| = m$, and $|A| = n+m$. Therefore $|A-B| = |A| - |B|$, as desired. This completes the proof.

3. **Claim**:

$$\sum_{m=1}^{n} \frac{1}{m(m+1)} = \frac{n}{n+1}$$

Proof: base case: We prove claim for the case $n = 1$:

$$\sum_{m=1}^{1} \frac{1}{m(m+1)} = \frac{1}{2} = \frac{1}{1+1}.$$

inductive hypothesis: Suppose that

$$\sum_{m=1}^{k} \frac{1}{m(m+1)} = \frac{k}{k+1}.$$

inductive step: Now we show the claim holds for $n = k+1$:

$$\begin{aligned}
\sum_{m=1}^{k+1} \frac{1}{m(m+1)} &= \sum_{m=1}^{k} \frac{1}{m(m+1)} + \frac{1}{(k+1)(k+2)} \\
&= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} \\
&= \frac{k(k+2)+1}{(k+1)(k+2)} \\
&= \frac{(k+1)^2}{(k+1)(k+2)} \\
&= \frac{k+1}{k+2}
\end{aligned}$$

By induction the proof is complete.

4. **Claim**: Let A_n be the sequence defined by $A_1 = 1$, and $A_{n+1} = A_n + 2^n$ for $n \geq 1$. Then

$$A_n = 2^n - 1$$

Proof: We prove this by induction on n. (Note that strong induction is not useful in this case since the definition of A_{n+1} only depends on A_n. If A_{n+1} was defined using A_{n-1} and A_n then strong induction would be useful.)

base case: We prove the claim for $n = 1$:

$$A_1 = 1 = 2^1 - 1$$

inductive hypothesis: Suppose the claim holds for $n = k$:

$$A_k = 2^k - 1.$$

inductive step: We prove the claim for $n = k + 1$:

$$\begin{aligned} A_{k+1} &= A_k + 2^k \\ &= (2^k - 1) + 2^k \\ &= 2(2^k) - 1 \\ &= 2^{k+1} - 1. \end{aligned}$$

By induction the proof is complete.

5. **(a)** Notice that f is invertible (with $f^{-1} = f$) so by Theorem 9.2.3 it is a bijection.

(b) In this case f is one-to-one since $x^2 = y^2$ and $x, y \geq 0$ then $x = y$. It is not onto since $f(x) \geq 0$ for all $x \in \mathbb{R}^{\geq}$, and therefore the strictly negative real numbers are not in the image of f.