INVARANTS OF FINITE GROUPS GENERATED BY
REFLECTIONS.*

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1. An invertible linear transformation of a finite dimensional vector
space $V$ over a field $K$ will be called a reflection if it is of order two and
leaves a hyperplane pointwise fixed. A group $G$ of linear transformations
of $V$ is a finite reflection group if it is a finite group generated by reflections.
The operations of $G$ extend to automorphisms of the symmetric algebra $S$
of $V$ by the rule $g(P)(x) = P(g^{-1}(x))$, $(P \in S, x \in V)$, and an element $P \in S$
such that $g(P) = P$ for all $g \in G$ is said to be an invariant of $G$. Our main
purpose in this note is to prove the theorem:

(A) Let $G$ be a finite reflection group in a $n$-dimensional vector-space
$V$ over a field $K$ of characteristic zero. Then the $K$-algebra $J$ of invariants
of $G$ is generated by $n$ algebraically independent homogeneous elements (and
the unit).

A vector space $A$ is graded by subspaces $A^i$, ($i$ positive integer), if it is
the direct sum of the $A^i$. The degree $d^0P$ of $P \in A$ is the smallest integer $j$
such that $P \in \sum_{i \leq j} A^i$; the elements of $A^i$ are the homogeneous elements of
degree $i$. When the $A^i$ are finite dimensional, the Poincaré series of $A$ in
the indeterminate $t$ is defined as

$$P_t(A) = \sum_{i \geq 0} \dim A^i \cdot t^i.$$ 

In particular $S$ is graded in the obvious way and $P_t(S) = (1 - t)^{-n}$. Let
$F$ be the ideal generated by the homogeneous elements of strictly positive
degrees in $J$. Then the grading of $S$ induces a grading of the quotient
space $S/F$. Since $F$ is invariant under $G$, the operations of $G$ in $S$ induce
automorphisms of $S/F$. We shall also prove:

(B) Let $I_1, \ldots, I_n$ be a minimal system of homogeneous generators
of $J$ and let $m_i$ be the degree of $I_i$, ($1 \leq i \leq n$). Then

$$P_t(S/F) = (1 - t)^{-n} \prod_{i=1}^{\ell} (1 - t^{m_i}).$$

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The product of the $m_i$ is equal to the order of $G$ and to the dimension of $S/F$. The natural representation of $G$ in $S/F$ is equivalent to the regular representation.

2. Two lemmas. In this paragraph, the characteristic $p$ of the infinite groundfield $K$ is allowed to be $\neq 0$ and $G$ denotes a finite reflection group in $V$ whose order $N$ is prime to $p$.\footnote{In this paper, we are primarily interested in the case $p = 0$, but Lemma 2 will be used in a forthcoming paper of A. Borel, to appear in Jour. Math. Pur. Appl.} To any element $P \in S$ we can then associate its average over $G$:

$$M(P) = \frac{1}{N} \sum_{g \in G} g(P).$$

**Lemma 1.** Let $U_1, \ldots, U_m$ be invariants of $G$ such that $U_1$ does not belong to the ideal generated in $J$ by $U_2, \ldots, U_m$. Let $P_i, (1 \leq i \leq m)$, be homogeneous elements of $S$ satisfying a relation $\sum P_i \cdot U_i = 0$. Then $P_1 \in F$.

If $d^0 P_1 = 0$, then it follows from the assumption and from the relation

$$M(P_1) \cdot U_1 + \cdots + M(P_m) \cdot U_m = 0$$

that $P_1 = M(P_1) = 0$. Assume now $d^0 P_1 > 0$ and the lemma to be true for all relations $\sum Q_i \cdot U_i = 0$ with homogeneous $Q_i$ and $d^0 Q_i < d^0 P_1$. Let $s$ be a reflection of $G$ leaving pointwise fixed a hyperplane with equation $L = 0$. Then $s(P_1) = P_1 = L \cdot Q_i, (Q_i \in S, i = 1, \ldots, m)$, and

$$Q_1 \cdot U_1 + \cdots + Q_m \cdot U_m = 0$$

whence, by induction, $Q_1 \in F$ or, otherwise said, $s(P_1) \equiv P_1 \mod F$; the group $G$ being generated by reflections, we have then $g(P_1) \equiv P_1 \mod F$ for any $g \in G$, whence $P_1 \equiv M(P_1) \mod F$; since $P_1$ is homogeneous of strictly positive degree, the same is true for $M(P_1)$; therefore $M(P_1) \in F$ and $P_1 \in F$.

**Lemma 2.** Assume $K$ to be a perfect field. Let $I_i, (1 \leq i \leq m)$, be homogeneous invariants which form an ideal basis of $F$,\footnote{This always exists since by the classical theorem for invariants of a finite group, $J$ is a finitely generated $K$-algebra.} with $m_i = d^0 I_i$, prime to $p$ for $i \leq r$. Then $I_1, \ldots, I_r$ are algebraically independent.

Let us suppose the lemma to be false and let $H(I_1, \ldots, I_r) = 0$ be a non trivial relation of minimal degree between $I_1, \ldots, I_r$ where $H(y_1, \ldots, y_r)$
is a polynomial in \( r \) letters \( y_i \). We may assume that there exists an integer \( h \) such that for any monomial \( y_1^{k_1} \cdots y_r^{k_r} \) of \( H \) we have
\[
k_1 \cdot m_1 + \cdots + k_r \cdot m_r = h.
\]

The partial derivatives \( \frac{\partial H}{\partial y_k} \) are not all zero, because otherwise (for \( p \neq 0 \), the only case for which it is not obvious), \( K \) being perfect, \( H \) would be the \( p \)-th power of a polynomial \( H^* \), and \( H^*(I_1, \ldots, I_r) = 0 \) would be a non trivial relation of strictly smaller degree. Set
\[
H_i = \frac{\partial H}{\partial y_k} (I_1, \ldots, I_r), \quad (1 \leq i \leq r);
\]
then \( H_1, \ldots, H_r \) are in \( J \) and not all zero; after a possible permutation of indices, we may assume that they belong to the ideal generated in \( J \) by the first \( s \) of them, but that none of \( H_1, \ldots, H_s \) belongs to the ideal generated by the other ones in \( J \). Set
\[
H_{s+1} = \sum_{f=1}^{i=s} V_{f,i} H_i.
\]

Let \( x_k, \ (1 \leq k \leq n) \), be coordinates in \( V \). Since
\[
\sum_{i=1}^{r} H_i \cdot (\partial I_i/\partial x_k) = 0, \quad (1 \leq k \leq n),
\]
we have by Lemma 1
\[
\partial I_i/\partial x_k + \sum_{j=1}^{j=r-s} V_{j,i} (\partial I_{s+j}/\partial x_k) \in F, \quad (1 \leq i \leq s; 1 \leq k \leq n)
\]
(the left hand sides are homogeneous in the \( x_k \) by the above remark on the monomials of \( H \)). Multiplying this relation by \( x_k \) and adding the relations thus obtained, we get
\[
m_i I_i + \sum_{j=1}^{j=r-s} V_{j,i} m_{s+j} I_{s+j} = \sum_{l=1}^{l=m} A_{i,l} I_l, \quad (1 \leq i \leq s).
\]
where the \( A_{i,l} \) are forms belonging to the ideal generated by \( x_1, \ldots, x_n \). For reasons of homogeneity, we have \( A_{i,1} = 0 \) if \( I_i \) is not of strictly lower degree than \( I_1 \); \( m_i \) being prime to \( p \) for \( i \leq r \), we see that \( I_i \) belongs to the ideal generated by the other \( I_j \), which is a contradiction. Thus \( I_1, \ldots, I_r \) are algebraically independent.

3. Proofs of Theorems (A) and (B). We assume again the ground-field to be of characteristic zero and denote as in Lemma 2 by \( I_1, \ldots, I_m \) homogeneous invariants of \( G \) forming an ideal basis of \( F \). By Lemma 2
they are algebraically independent, whence also $m \leq n$. Using averages over $G$, it is readily seen by induction on the degree that the unit and the $I_i$ generate $J$ and thus, to finish the proof of (A), there remains to show that $m \geq n$.

Let $x_1, \ldots, x_n$ be coordinates in $V$ and let $K(x)$ be the field of rational functions in the $x_i$. It is acted upon in a natural way by $G$ and we denote by $L$ the subfield of elements invariant under $G$. Then $K(x)$ is a Galois extension of $L$, with Galois group $G$ and $L$ has also transcendence degree $n$ over $K$. On the other hand, $G$ being finite, every invariant in $K(x)$ is classically the quotient of two invariant polynomials; thus $L = K(J)$ is generated by the $I_i$ and $m \geq n$.

**Lemma 3.** Let $P_1, \ldots, P_s$ be homogeneous elements of $S$ whose residue classes mod $F$ are linearly independent over $K$ in $S/F$. Then $P_1, \ldots, P_s$ are linearly independent over $K(J)$.

Let $V_1 \cdot P_1 + \cdots + V_s \cdot P_s = 0$ be a relation with $V_i \in K(J), (1 \leq i \leq s)$. We have to prove that $V_i = 0$ for all $i$ and it is enough to consider the case where the $V_i$ are homogeneous elements of $J$ such that $d^0V_i + d^0P_i$ is equal to a constant $h$ independent of $i$.

By the degree of the monomial $I_1^{k_1} \cdots I_n^{k_n}$ we mean its degree as element of $S$, i.e. $k_1m_1 + \cdots + k_nm_n$. Let $S_j, (j = 1, 2, \ldots)$, be the different monomials in the $I_i$ arranged by increasing degrees, with $S_1 = 1$. We have

$$V_i = \sum_{j \geq 0} k_{ij}S_j, \quad (k_{ij} \in K, k_{ij} = 0 \text{ for } d^0V_i \neq d^0S_j, (1 \leq i \leq n)),$$

and our relation may be written

$$\sum_{j \geq 0} W_j \cdot S_j = 0, \quad (W_j = \sum_{i=1}^{k_{ij}} k_{ij}P_i),$$

where $W_j$ is homogeneous, of degree equal to $h - d^0S_j$. Assume that $k_{ij} = 0$ for $1 \leq i \leq s$ and $j < t$. Since by Theorem A the monomial $S_t$ does not belong to the ideal generated in $J$ by the $S_j$ with $j > t$, we have by Lemma 1 $W_t \in F$ and the hypothesis gives then $k_{it} = 0$ for $i = 1, \ldots, s$. This proves by induction on $j$ that $k_{ij} = 0$ for all $i, j$, and the lemma.

We now come to the proof of (B). The field $K(x)$ being a normal extension of $K(J)$ with Galois group $G$, it has degree $N$ over $K(J)$, hence the dimension of $S/F$ over $K$ is finite. Let $A_1, \ldots, A_q$ be homogeneous polynomials whose residue classes mod $F$ form a basis of $S/F$. By induction on the degree we see that every $P \in S$ may be expressed as linear combination

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of the $A_i$ with coefficients in $J$, and this expression is unique in view of Lemma 3. Hence

$$P_t(S) = P_t(S/F) \cdot P_t(J);$$

but $P_t(S) = (1-t)^{-n}$ and Theorem A gives $P_t(J) = \prod_{i=1}^{m_1} (1-t^{m_1})^{-1}$, whence the first assertion of (B). We may also write

$$P_t(S/F) = \prod_{i=1}^{m_1} (1 + t + t^2 + \cdots + t^{m_1-1})$$

and, setting $t = 1$, we get $\text{dim. } S/F = m_1 \cdots m_n$. Since every element of $K(x)$ may be written as the quotient of a polynomial by an invariant polynomial, it also follows from the above and Lemma 3 that the $A_i$ form a basis of $K(x)$ over $K(J)$, whence $N = \text{dim. } S/F$.

We have for $g \in G$

$$g(A_i) = \sum_{j=1}^{N} a_{ij}(g) A_j,$$

where the $a_{ij}(g)$ are homogeneous elements of $J$ and where $a_{ij}(g) \in K$ by homogeneity. The matrices $(a_{ij}(g))$ describe the natural representation of $G$ in $K(x)$, considered as vector space over $K(J)$. If we reduce the coefficients mod $F$ we get the natural representation of $G$ in $S/F$, considered as vector space over $K$; this reduction does not affect the diagonal coefficients, hence both representations have the same character and are equivalent. But $G$ is the Galois group of the normal extension $K(x)$ of $K(J)$, so that the former representation is equivalent to the regular representation, which proves the last statement of (B).

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