Solutions to some of the additional problems

In the problems below the term Taylor series at $z_o$ of $f(z)$ means the power series

$$\sum_{m=0}^{\infty} a_n (z - z_o)^n$$

that converges to $f(z)$ at the open disk with center $z_o$ of radius the radius of convergence of the series.

1. Prove that if $f(z)$ is analytic on the punctured disk \( \{ z \mid 0 < |z - z_o| < r \} \) and if $f$ is continuous at $z = z_o$ then $f$ is analytic on the entire disk.

Solution: If $f$ is continuous at $z_o$ and analytic for $0 < |z - z_o| < r$ then by the definition of continuity we have that if $\varepsilon > 0$ is given then there exists $\delta > 0$ such that if $|z - z_o| < \delta$ then $|f(z) - f(z_o)| < \varepsilon$. We may fix $\varepsilon$. Let $s$ be a choice of $\delta$ with $0 < s < r$. If $|z - z_o| < s$ then

$$|f(z)| = |f(z_o) + (f(z) - f(z_o))| \leq |f(z_o)| + |f(z) - f(z_o)|$$

$$< |f(z_o)| + \varepsilon.$$ 

This implies that $f(z)$ is bounded in a punctured neighborhood of $z_o$ so the removable singularities theorem implies that $f$ is analytic at $z_o$.

2. Let $C$ be the circle of radius 1 and center 1 oriented counter clockwise. For $n = 1, 2, 3, ...$ calculate

$$\int_C \left( \frac{z}{z - 1} \right)^n dz.$$ 

Solution: We write

$$\left( \frac{z}{z - 1} \right)^n = \left( \frac{1 + (z - 1)}{z - 1} \right)^n =$$

$$\frac{\sum_{j=0}^{n} \binom{n}{j} (z - 1)^j}{(z - 1)^n} = \sum_{j=0}^{n} \binom{n}{j} (z - 1)^{j-n}.$$ 

The coefficient of $\frac{1}{z-1}$ is thus $\binom{n}{n-1} = n$. Thus $\text{Res}_{z=1} \left( \frac{z}{z - 1} \right)^n = n$. Hence the residue theorem implies that

$$\int_C \left( \frac{z}{z - 1} \right)^n dz = 2\pi ni.$$
3. Calculate the following integrals with $C$ the unit circle oriented counter clockwise.
   a) $\int_C \frac{e^{2iz}}{z^2} \, dz$.
   b) $\int_C \frac{\sin z}{z^5} \, dz$.

Solutions: a) Consider the expansion
   $$\frac{e^{2iz}}{z^2} = \frac{1}{z^2} \sum \frac{(2iz)^n}{n!} = \frac{1}{z^2} \left( 1 + \frac{2iz}{1!} + \frac{(2iz)^2}{2!} + \ldots \right).$$

Thus the coefficient of $\frac{1}{z}$ is $2i$ so the residue theorem says that the first integral is $2\pi i (2i) = -4\pi$.

   b) First version. As in a)
   $$\frac{\sin z}{z^5} = \frac{1}{z^5} \left( z - \frac{z^3}{3!} + \frac{z^5}{5!} + \text{higher powers} \right).$$

We therefore see that only even powers come into the expansion. Thus the coefficient of $\frac{1}{z}$ is 0. The residue theorem implies that the integral is 0.

Second version: The function $f(z) = \frac{\sin z}{z^5}$ satisfies $f(-z) = f(z)$ so only even powers can appear in its Laurent expansion at 0. Thus the residue is 0.

4. Let $\text{Log}(z)$ denote the principal branch of the logarithm. Prove that its Taylor series centered at $z_0 = 1$ is
   $$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{(z-1)^n}{n}.$$

(Hint: Show that the radius of convergence of the series is 1 and calculate its derivative. Compare the derivative with that of $\text{Log}(z)$.)

Solution: The coefficient of $(z-1)^n$ is $a_n = \frac{(-1)^{n-1}}{n}$ for $n = 1, 2, \ldots$ So
   $$\left| \frac{a_n}{a_{n+1}} \right| = \frac{n+1}{n}.$$ Hence $\lim_{n \to \infty} \frac{n+1}{n} = 1$. The radius of convergence is therefore 1. The derivative for $|z-1| < 1$ is
   $$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n(z-1)^{n-1}}{n} = \sum_{n=0}^{\infty} (-1)^n (z-1)^n = \frac{1}{1 + (z-1)} = \frac{1}{z}.$$

We note that the set $\{ z | |z-1| < 1 \}$ is in the domain of $\text{Log}$ (no nonpositive real numbers are in the set). Also $\frac{d}{dz} \text{Log}(z) = \frac{1}{z}$ in its domain. This implies that the derivative of $\text{Log}(z) - f(z)$ is 0 for $|z-1| < 1$. Hence $\text{Log}(z) - f(z) = c$ a constant for $|z-1| < 1$. But $f(1) = 0$ and $\text{Log}(1) = 0$ so $c = 0$.
5. Let $C_\rho$ denote the circle of radius $0 < \rho < 1$ and center 1 oriented counter clockwise. Calculate
\[
\int_{C_\rho} \frac{\log(z)dz}{(z - 1)^n}
\]
for all $n$.
Solution: In exercise 4 we computed the Taylor series of $\log(z)$ at $z = 1$. The coefficient of $(z - 1)^k$ is 0 if $k \leq 0$ and $\frac{(-1)^{k-1}}{k}$ if $k > 0$. If $n < 0$ then the function $\frac{\log(z)}{(z - 1)^n}$ is analytic on $|z - 1| < 1$ so the integral is 0. If $n = 1$ then since $\log(1) = 0$ we have $\lim_{z \to 1} \frac{\log(z)}{z-1}$ is the derivative of $\log(z)$ at $z = 1$ that is, 1. Hence we can apply the removable singulatities theorem to see that the function defined to be $\frac{\log(z)}{z-1}$ if $z \neq 1$ and 1 at $z = 1$ is analytic on the domain of Log. So if $n = 1$ then the integral is 0. If $n > 1$ the coefficient of $\frac{1}{z-1}$ in the Laurant series of $\frac{\log(z)}{(z - 1)^n}$ is
\[
\frac{(-1)^{n-2}}{n-1} = (-1)^n \frac{n}{n-1}
\]
so the integral is
\[
\frac{(-1)^n 2\pi i}{n-1}.
\]

6. Define
\[
f(z) = \frac{1}{2i} \log \left( \frac{1 + iz}{1 - iz} \right).
\]
Find the domain of $f$ and show that $\tan(f(z)) = iz$ (there was an error here in the original-sorry!) Show that the Taylor series at 0 of $f(z)$ is
\[
\sum_{n=0}^{\infty} (-1)^n \frac{z^{2k+1}}{2k + 1}.
\]
(Hint: $\log\left( \frac{1 + iz}{1 - iz} \right) = \log(1 + iz) - \log(1 - iz)$. See problem 4.)
Solution: The domain of Log is all $z$ that are not nonpositive real numbers. We first check when $g(z) = \frac{1 + iz}{1 - iz}$ is real. That is when
\[
g(z) - \overline{g(z)} = 0.
\]
We write this out
\[
\frac{1 + iz}{1 - iz} - \frac{1 - iz}{1 + iz} = \frac{(1 + iz)(1 + iz) - (1 - iz)(1 - iz)}{(1 - iz)(1 + iz)}
\]
\[
= \frac{(1 + i(z + |z|^2) - (1 - i(z + |z|^2))}{(1 - iz)(1 + iz)}
\]
\[
= \frac{2i(z + |z|^2)}{(1 - iz)(1 + iz)}.
\]

Thus the expression is real only if \(z = iy\) with \(y\) real. If \(z = iy\) then
\[
g(z) = \frac{1 - y}{1 + y}.
\]

This expression is \(\leq 0\) if and only if \(\frac{1 - y}{1 + y} \leq 0\). The denominator is positive if \(y < 1\) and in this case the denominator is nonpositive if \(y < -1\). The numerator is negative of \(y > 1\) and in this case the denominator is always positive. Thus the expression is negative if and only if \(|y| > 1\). Thus the domain of the function is the set of all \(z\) that are not of the form \(iy\) with \(|y| \geq 1\).

To prove the formula we note that \(\tan w = e^{iw} - e^{-iw} \frac{e^{iw}}{e^{iw} + e^{-iw}}.\) We substitute \(w = \frac{1}{2i} \Log(\frac{1 + iz}{1 - iz})\) and get
\[
e^{\frac{1}{2} \Log(\frac{1 + iz}{1 - iz})} - e^{-\frac{1}{2} \Log(\frac{1 + iz}{1 - iz})}
\]
\[
e^{\frac{1}{2} \Log(\frac{1 + iz}{1 - iz})} + e^{-\frac{1}{2} \Log(\frac{1 + iz}{1 - iz})}
\]

multiplying numerator and the denominator by \(e^{\frac{1}{2} \Log(\frac{1 + iz}{1 - iz})}\) we have
\[
e^{\Log(\frac{1 + iz}{1 - iz})} - 1\]
\[
e^{\Log(\frac{1 + iz}{1 - iz})} + 1 \]
\[
= \frac{1 + iz}{1 - iz} - 1\]
\[
= \frac{1 + iz}{1 - iz} + 1\]
\[
= \frac{2iz}{2} = iz.
\]

Finally if we substitute into the series we have
\[
\Log(1 + iz) - \Log(1 - iz) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(1 + iz)^n - 1}{n} - \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(1 - iz)^n - 1}{n}
\]
\[
= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(iz)^n}{n} - \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(-iz)^n}{n}.
\]
We note that the even powers in the series cancel out and so we have
\[ 2 \sum_{n=0}^{\infty} (-1)^{2n+1} \frac{(iz)^{2n+1}}{2n+1} = 2 \sum_{n=0}^{\infty} i^{2n+1} \frac{z^{2n+1}}{2n+1} = 2i \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{2n+1}. \]

7. Define \( f(z) \) to be
\[ f(z) = \frac{z}{e^z - 1} \]
for \( z \neq 0 \) and \( f(0) = 1 \). Show that \( f(z) \) is analytic at 0. Find the radius of convergence of its Taylor series
\[ \sum_{n=0}^{\infty} a_n z^n \]
at \( z = 0 \). Calculate \( a_0, a_1 \) and \( a_2 \). (Hint: Use Exercise 1 to prove that the function is analytic. You will have to use L’Hospital’s rule to calculate the coefficients.)

Solution: \( \lim_{z \to 1} \frac{e^z - 1}{z} = 1 \). Thus if \( g(z) = \frac{e^z - 1}{z} \) for \( z \neq 0 \) and \( g(z) = 1 \) for \( z = 0 \) Exercise 1 implies that \( g(z) \) is analytic at 0. Thus \( f(z) = \frac{1}{g(z)} \) is analytic at 0. The zeros of the denominator of \( f(z) \) are at the points \( 2\pi ki \) for \( k \) an integer. Since it is analytic at \( z = 0 \) we see that the smallest absolute value of a singularity is \( 2\pi \). Thus \( 2\pi \) is the radius of convergence of its Taylor series. We have already calculated \( a_0 = f(0) = 1 \). To calculate \( a_1, a_2 \) we must calculate \( f'(0) \). The quotient rule gives for \( z \neq 0 \)
\[ f'(z) = \frac{(e^z - 1) - z e^z}{(e^z - 1)^2}. \]

To take the limit as \( z \to 1 \) we use L’Hospital’s rule an take the derivative of the numerator and denominator
\[ f'(0) = \lim_{z \to 0} \frac{-ze^z}{2e^z(e^z - 1)} = -\frac{1}{2}. \]

We next look at \( f''(z) \) which is given by
\[ \frac{-ze^z(e^z - 1)^2 - 2e^z(e^z - 1)((e^z - 1) - ze^z)}{(e^z - 1)^4} = -\frac{e^z(2 + e^z(-2 + z) + z)}{(e^z - 1)^3}. \]

If we take the third derivative of the numerator and denominator we get
\[ \lim_{z \to 0} f''(z) = \lim_{z \to 0} \frac{5e^z - 4e^{2z} + ze^z + 8ze^z}{3e^z - 24e^{2z} + 27e^{3z}} = \frac{1}{6}. \]
Thus \( a_0 = 1, a_1 = -\frac{1}{2}, a_2 = \frac{1}{2} f''(0) = \frac{1}{12}. \)

8. What is wrong with the following argument? Let \( f(z) \) be a polynomial. Then since

\[
\lim_{z \to \infty} \frac{f(z)}{e^{z^2}} = 0
\]

we see that the function

\[
\frac{f(z)}{e^{z^2}}
\]

is bounded. Liouville’s theorem implies that it is a constant. Since the limit is 0 we see that the constant must be 0. Thus

\[
\frac{f(z)}{e^{z^2}} = 0 \text{ for all } z.
\]

Hence \( f(z) = 0 \) for all \( z \). We conclude that all polynomials are 0.

Solution: This problem is bogus. We must show that initial premise is false for at least one polynomial. If \( u = 1 + i \sqrt{2} \) then \( u^2 = i \). Thus if \( z = u \sqrt{2k\pi} \) then \( e^{z^2} = e^{u^2 2k\pi} = e^{2k\pi i} = 1 \). Thus the initial premise implies that

\[
\lim_{k \to \infty} \frac{f(u \sqrt{2k\pi})}{1} = 0.
\]

This is false for the constant polynomial \( f(z) = 1 \).

9. Let \( C \) be the circle of radius \( r > 0 \) with center \( z_0 \). Oriented counterclockwise Let \( a \) be a complex number not on the circle calculate

\[
\frac{1}{2\pi i} \int_C \frac{dz}{z-a}.
\]

(Hint: \( a \) can both inside and outside of the circle.) Suppose now that \( C \) is a simply closed oriented curve. Show that the integral takes on exactly two values. What do the values have to do with the “inside” and “outside” of the curve?

Solution: Let us assume that it is oriented counterclockwise. The indicated number is the winding about the point \( a \). This was explained in class.