Let G be a commutative group. We define $\widehat{G}$ to be the set of all group homomorphisms from G as a group under addition to $\mathbb{C}^{\times} = \mathbb{C} - \{0\}$. Such homomorphisms are called characters. We have defined a group structure on $\widehat{G}$ as follows: If $\chi_1, \chi_2 \in \widehat{G}$ then

$$(\chi_1\chi_2)(x) = \chi_1(x)\chi_2(x).$$

The identity in the group is 1 which is our notation for the function $\chi(x) = 1$ for all $x \in G$.

If $G = R_m = \mathbb{Z}/m\mathbb{Z}$, for $m > 0$ then every character of R_m is of the form

$$\chi_j(x) = e^{\frac{2\pi i j x}{m}}.$$

With $j = 0, 1, ..., m-1$ and x is taken as an element of $\{0, 1, ..., m-1\}$. We note that χ_j depends only on its equivalence class modulo m.

If $y \in R_m$ define a function $\eta_y : \widehat{R_m} \to \mathbb{C}^{\times}$ by $\eta_y(\chi) = \chi(y)$.

1) Prove that η_y defines a character of $\widehat{R_m}$.

2) Prove that if η is a character of $\widehat{R_m}$ then there exists $y \in R_m$ such that $\eta = \eta_y$.

3) Prove that the mapping $\varphi : R_m \to \widehat{R_m}$ is a group isomorphism (here $R_m is$ considered to be a group under addition).

4) Prove that if $\chi \in \widehat{R_m}$ then

$$\sum_{x \in R_m} \chi(x) = \left\{ \begin{array}{c} m \text{ if } \chi = 1 \\ 0 \text{ if } \chi \neq 1 \end{array} \right. .$$

(Hint: We may assume that $\chi = \chi_j$ for some j.)

5) Prove that if $x \in R_m$ then

$$\sum_{\chi \in \widehat{R_m}} \chi(x) = \left\{ \begin{array}{c} m \text{ if } x = 0 \\ 0 \text{ if } x \neq 0 \end{array} \right. .$$

If $f : R_m \to \mathbb{C}$ define $\mathcal{F}(f) : \widehat{R_m} \to \mathbb{C}$ by

$$\mathcal{F}(f)(\chi) = \frac{1}{\sqrt{m}} \sum_{y \in R_m} \chi(y)^{-1} f(y).$$

6) Prove that

$$f(x) = \frac{1}{\sqrt{m}} \sum_{\chi \in \widehat{R_m}} \chi(x)\mathcal{F}(f)(\chi)$$

for all $f : R_m \to \mathbb{C}$ and all $x \in R_m$. (Hint: Write out the right hand sum as a double sum and show that the double sum can be written

$$\frac{1}{m} \sum_{\chi \in \widehat{R_m}} \sum_{y \in R_m} \chi(x-y)f(y).$$

Make the change of variables $z = x - y$ so $y = x - z$ and write the sum

$$\frac{1}{m} \sum_{z \in R_m} \Big(\sum_{\chi \in \widehat{R_m}} \chi(z)\Big) f(x-z).$$

Now use the results above.)

If $f : R_m \to \mathbb{C}$ then we can define a new function $\widehat{f}(j) = \mathcal{F}f(\chi_j)$ mapping R_m to $\mathbb{C}$.

7) Write out the formula for $\widehat{f}(j)$.

8) Prove that $\widehat{\widehat{f}}(x) = f(-x)$.