

Extra Exercises for 1/26

1. Show that if  $p$  is an odd prime and  $a \in \mathbb{Z}$  is such that  $p \nmid a$  and  $k \geq 1$  then

$$a^{p^{k-1}(p-1)} \equiv 1 \pmod{p^k}.$$

2. Prove under the hypotheses in 1. that

$$\frac{a^{p^{k-1}(p-1)} - 1}{p^k} \equiv \frac{a^{p-1} - 1}{p} \pmod{p}.$$

3. Let  $\zeta = e^{2\pi i/3}$ . Find  $f(t) = a_0 + a_1t + a_2t^2 + t^3$  with  $a_i \in \mathbb{Z}$  for  $i = 0, 1, 2, 3$  such that  $f(1 + \zeta) = 0$ . Is there another monic (coefficient of the coefficient of the highest power of  $t$ ) polynomial of degree at most 3 with integer coefficients,  $g(t)$ , such that  $g(1 + \zeta) = 0$ ?