Extra Problems 3/3/09

1. Prove that $\pi(m) \leq \frac{m}{3}$ for $m \in \mathbb{Z}$, $m \geq 33$.

2. Let $\sigma(x)$ be the number of integers $1 \leq n \leq x$ such that n is a square prove that

$$\lim_{x\infty+\infty} \frac{\sigma(x)}{\pi(x)} = 0.$$

3. Let p_n be the n^{th} prime (in pari gp $p_n = prime(n)$). Calculate $\frac{p_n}{n\log(n)}$ for $n = k(1000)$, $k = 20, 40, 60, 80, 100$ (you will have to increase prime limit). What would you guess about

$$\lim_{n\to\infty} \frac{p_n}{n \log n}?$$

4. Let p_n be as in problem 3. Use the fact that $\pi(p_n) = n$ to prove that there exist constants $A, B > 0$ such that

$$An \log n \leq p_n \leq Bn \log n.$$

(Hint: Show that there is a constant, C, such that $p_n \leq Cn^2$.)