Extra Homework for 10/13

1. Find all integers X, Y, Z satisfying the system of equations

$$2X + 3Y + 5Z = 0$$

$$3X + 5Y + 7Z = 0.$$

2. Show that if $m, a, b \in \mathbb{Z}_{>0}$ and if $\gcd(a, b) = 1$ then the there exist integers x, y such that

$$\frac{m}{ab} = \frac{x}{a} + \frac{y}{b}.$$

The next 4 problems are designed to prove that there are infinitely many primes of the form $4k + 3$ with $k \in \mathbb{N}$.

3. Show that $n \in \mathbb{Z}$ is of the form $n = 4k + 3$ with $k \in \mathbb{N}$ if and only if it is of the form $4l - 1$ with $l \geq 1$.

4. Show that if $n_i = 4k_i + 1$ with $k_i \in \mathbb{N}$ for $i = 1, ..., m$. Then $n_1 n_2 \cdots n_m = 4k + 1$ for some $k \in \mathbb{N}$.

5. Show that if p is a prime with $p \neq 2$ then p is of the form $4k + 1$ with $k \in \mathbb{Z}_{>0}$ or of the form $4k + 3$ with $k \in \mathbb{N}$.

6. Show that if $N \geq 2$ and $p_1 < p_2 < ... < p_N$ are primes with $p_i = 4k_i + 3$ with $k_i \in \mathbb{N}$ then

$$n = 4p_1 p_2 \cdots p_N - 1$$

is divisible by a prime q with q of the form $4k + 3$ and q is not in the set $\{p_1, ..., p_N\}$.

(Hint: n is either prime or composite. If it is composite and only divisible by primes of the form $4k + 1$ then we have a contradiction (to what?).)