Extra problems

1. Let

\[ \sigma(t) = \frac{t + i}{it + 1} \]

for \( t \) a real number. Show that if \( a < b \) then the curve parametrized by \( \sigma(t) \) for \( a < t < b \) follows the unit circle in the clockwise direction. Show that if \( |z| = 1 \) and \( z \neq -i \) then there is a unique real number \( t \) such that \( \sigma(t) = z \).

2. Let \( f(z) \) be an analytic function on the open set \( U \). Show that if \( a \in U \) and \( f(a) = 0 \) then there exists \( g(z) \) an analytic function such that \( f(z) = (z - a)g(z) \).

3. Calculate the following contour integrals

\[ \int_C \frac{\sinh z}{z^4} dz \]

for contours \( C \) as below:

4. The purpose of this problem is to calculate the integral

\[ \int_0^\infty \frac{\sin x}{x(1 + x^2)} dx. \]

You get partial credit for each of the steps that you complete. In some cases it is a calculation that must be done. In others an explanation must be given.
a) First show that if $0 < r < R$ then
\[ \int_{r}^{R} \frac{\sin x}{x(1 + x^2)} \, dx = \frac{1}{2i} \int_{-R}^{-r} \frac{e^{ix}}{x(1 + x^2)} \, dx + \frac{1}{2i} \int_{r}^{R} \frac{e^{ix}}{x(1 + x^2)} \, dx. \]

b) Set
\[ f(z) = \frac{e^{iz}}{z(1 + z^2)}. \]

Consider the contour $C_{r,R}$

where the outer path is the semicircle of radius $R$ and the inner is the semicircle of radius $r$. show that
\[ \int_{C_{r,R}} f(z) \, dz = \frac{-\pi i}{e}. \]

using the residue theorem.

c) Prove that
\[ f(z) - \frac{1}{z} \]
has a removable singularity at $z = 0$. Use this to show that
\[ \lim_{r \to 0} \int_{C_r} \left( f(z) - \frac{1}{z} \right) \, dz = 0 \]
where $C_r$ is the part of $C_{r,R}$ going from $-r$ to $r$.

d) Show that
\[ \int_{C_r} \frac{dz}{z} = -\pi i. \]
e) In this part I will be doing most of the work for you. You must explain why I am correct. Write out $\int_{C_{r,R}} f(z) dz$ as a sum

$$
\int_{-R}^{-r} \frac{e^{ix}}{x(1 + x^2)} \, dx + \int_{C_r} f(z) \, dz + \int_{r}^{R} \frac{e^{ix}}{x(1 + x^2)} \, dx + \int_{C_R} f(z) \, dz
$$

where $C_R$ is the part of the contour from $R$ to $-R$ along the outer circle. Write this as

$$
\int_{-R}^{-r} \frac{e^{ix}}{x(1 + x^2)} \, dx + \int_{C_r} \frac{df(z)}{z} + \int_{r}^{R} \frac{e^{ix}}{x(1 + x^2)} \, dx - \pi i
$$

Conclude that

$$
\int_{-R}^{-r} \frac{e^{ix}}{x(1 + x^2)} \, dx + \int_{C_r} \frac{df(z)}{z} + \int_{r}^{R} \frac{e^{ix}}{x(1 + x^2)} \, dx = (\pi e - \pi)i - \int_{C_r} \frac{df(z)}{z} + \int_{C_C} f(z) \, dz.
$$

f) Show that

$$
\lim_{R \to \infty, r \to 0} \left( \int_{C_r} \frac{df(z)}{z} + \int_{C_C} f(z) \, dz \right) = 0.
$$

g) Now put all the steps together to calculate

$$
\int_0^{\infty} \frac{\sin x}{x(1 + x^2)} \, dx = \lim_{r \to 0, R \to \infty} \frac{1}{2i} \left( \int_{-R}^{-r} \frac{e^{ix}}{x(1 + x^2)} \, dx + \int_{r}^{R} \frac{e^{ix}}{x(1 + x^2)} \, dx \right).
$$