To give an idea of what this course will be about we look at the following well known ~~ex~~ example.
Let $V = M_n(\mathbb{C})$ be the $n \times n$ matrices over $\mathbb{C}$. We have $G = GL(n, \mathbb{C})$ acting on it via $g \cdot X = gXg^{-1}$.

As we know the similarity classes ($G \cdot X = \{gXg^{-1} \mid g \in G\}$) i.e. orbits are parametrized by Jordan Canonical form.

That is if $J_k = \begin{array}{c c} & k \\ k & \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{bmatrix} \end{array}$

Then every X is similar to block a unique matrix of the form

$$\Phi C = \begin{bmatrix} \boxed{\lambda_1 I_{n_1} + J_{n_1}} & & & 0 \\ & \boxed{\lambda_2 I_{n_2} + J_{n_2}} & & \\ & & \ddots & \\ 0 & & & \boxed{\lambda_k I_{n_k} + J_{n_k}} \end{bmatrix}$$

with $n_1 \geq \cdots \geq n_k > 0$ $\sum n_i = n$.

We note that if

$$\varphi_n(z) = \begin{bmatrix} z^n & & & 0 \\ & z^{n-1} & & \\ & & \ddots & \\ 0 & & & z \end{bmatrix}$$

Then $\varphi_n(z) J_n \varphi_n(z)^{-1} = z J_n$.

Thus if

$$\varphi(z) = \begin{bmatrix} \varphi_1(z) & & & 0 \\ & \varphi_2(z) & & \\ & & \ddots & \\ 0 & & & \varphi_k(z) \end{bmatrix}$$

Then $\varphi(z) C \varphi(z)^{-1} =$

has blocks

$$\lambda_j I_{n_j} + z J_{n_j}$$

This implies that

$$\lim_{z \to 0} \varphi(z) C \varphi(z)^{-1} =$$

$$\begin{bmatrix} \lambda_1 I_{n_1} & & 0 \\ & \ddots & \\ 0 & & \lambda_k I_{n_k} \end{bmatrix},$$

If $X \in M_n(\mathbb{C})$

$$\det(tI - X) = t^n - \operatorname{tr} X\, t^{n-1} + D_2(X)\, t^{n-2} \mp \cdots (-1)^n \det X.$$

$$= \cancel{t^n}\; t^n \cancel{+ \sum a_j(X) t^j} + \sum (-1)^{n-j} \cancel{a_j(X)}\, D_{n-j}(X)\, t^j$$

We note that if $X = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$

Then $D_j(X) = \sum_{1 \le i_1 < \cdots < i_j \le n} \lambda_{i_1} \cdots \lambda_{i_j} =$

$e_j(\lambda_1, \dots, \lambda_n)$ the j^{th}

elementary symmetric function.

We also note that

if $t^n + \sum_{j=0}^{n-1} (-1)^{n-j} a_{n-j} t^j$ is

a polynomial in t then

up to order the a_j determine

the roots $\cancel{\lambda_1, \dots, \lambda_n}$ counting

~~mult.~~ $\underbrace{\lambda_1, \dots, \lambda_1}_{n_1} \underbrace{\lambda_2, \dots, \lambda_2}_{n_2} \dots$

mult $n_1 \ge \cdots \ge n_k > 0$ $\sum n_i = n$.

and conversely the roots determine

$a_0, \dots, a_{n-1}$, $\cancel{a_{n-j}}$ $a_j = e_j(\lambda_1, \dots)$

We therefore see that
the polynomials D_j in x_{ij}, $X = [x_{ij}]$,
uniquely determine ~~the~~ and are
determined by the eigenvalues
of X counting multiplicity.

$z_1, \dots, z_n \in \mathbb{C}$ are given
let $W_z = \{ X \in M_n(\mathbb{C}) \mid D_j(X) = z_j \}$,
then W_z is a finite union
of orbits such that ~~the~~
~~in~~ the diagonal part of the
Jordan canonical form of the
orbit ~~is~~ has diagonal entries
the roots of $\cancel{Z}\; t^n + \sum_{j=0}^{n-1} (-1)^{n-j} D_j(X) t^j$,

in some order.

We note that if $X \in W_Z$ then there exists $\varphi : \mathbb{C}^\times \to GL(n, \mathbb{C})$ polynomial homomorphism such that $\lim_{t \to 0} \varphi(t) X \varphi(t)^{-1} = Z$ is the unique diagonal element of W_Z up to order.

This implies that the orbit of ~~Z~~ a diagonal element in W_Z is in the closure of every orbit in W_Z.

On the other hand we will see that the closure of an orbit always contains a unique closed orbit.

This implies that the orbits of the diagonal elements are the closed orbits.

Let us abstract what we have here. G is a nice ~~[illegible]~~ group acting linearly on a finite dim vector space V.

$$V = \bigcup_{v \in V} G \cdot v$$

a union of orbits.

$\mathcal{O}(V)$ polynomial functions on V. $\mathcal{O}(V)^G$ the ~~[illegible]~~ G-invariant polynomial functions on V.

~~[illegible] $f_1, f_2, \ldots, f_m$ a set of generators~~

If $v \in V$ then ~~[illegible]~~ $\exists\ U_v = \{ w \in V \mid \varphi(w) = \varphi(v),\ \varphi \in \mathcal{O}(V)^G \}$.

Then

(1) U_v contains a unique closed orbit.

(2) ~~the closed G centralized~~ If ~~$v \in U$~~ $w \in U_v$ there

exists ~~a unique~~ $\varphi : \mathbb{C}^\times \to G$

a nice group homomorphism

such that ~~G~~ $\lim_{z \to 0} \varphi(z) w$ ~~[illegible]~~

the ~~G~~ closed orbit.

That is closed orbits are

~~of~~ the only ~~vs~~ stable ones.

In particular,
(Hilbert-Mumford),

(3) ~~$\varphi(z)$~~ $\cdot$ ~~f~~$(v) = 0$ all $f \in \mathcal{O}(V)^G$

then

$f(0) = 0 \iff \exists\ \varphi : \mathbb{C}^\times \to G$

with $\lim_{z \to 0} \varphi(z) \cdot v = 0$