# 1 Some multilinear algebra.

Let $F$ be a field and let $V$ be a vector space over $F$. Let $T(V)$ be the tensor algebra on $V$. In other words, $T(V)$ is the direct sum of the vector spaces $\bigotimes^k V$ and the multiplication is given by juxtaposition. That is,

$$(v_1 \bigotimes v_2 \bigotimes \cdots \bigotimes v_k) \bigotimes (w_1 \bigotimes w_2 \bigotimes \cdots \bigotimes w_l) =$$

$$v_1 \bigotimes v_2 \bigotimes \cdots \bigotimes v_k \bigotimes w_1 \bigotimes w_2 \bigotimes \cdots \bigotimes w_l.$$

Let $j : V \to T(V)$ be the map $v \to v \in \bigotimes^1 V$. Then this algebra is characterized up to automorphism by the universal mapping property: If $\rho : V \to \mathcal{A}$ is an $F$-linear map of $V$ into the associative algebra (not necessarily commutative) with unit,$\mathcal{A}$,over $F$ then there is a unique algebra homomorphism (taking the identity to the identity), $\widetilde{\rho} : T(V) \to \mathcal{A}$ such that $\widetilde{\rho} \circ j = \rho$. We now consider the quotient algebra

$$\bigwedge V = T(V)/ \sum_{v \in V} T(V)v \bigotimes vT(V).$$

Let $\bigwedge^k V$ be the image of $\bigotimes^k V$ in this quotient. We write for $u, v \in \bigwedge V$ the quotient algebra multiplication as $u \bigwedge v$. We define a map of the $k$-fold product

$$\sigma : V \times V \times \cdots \times V \to \bigwedge^k V$$

by $\sigma(v_1, ..., v_k) = v_1 \bigwedge \cdots \bigwedge v_k$. Here we identify $j(v) + \sum_{v \in V} T(V)v \bigotimes vT(V)$ with $v$ for $v \in V$. Let $W$ be a vector space over $F$. We say that a map $\lambda : V \times V \times \cdots \times V \to W$ (here the product is $k$-fold) is $k$-multilinear if for each fixed choice of $v_1, ..., v_{k-1}$ and $1 \leq i \leq k$, the map

$$v \longmapsto \lambda(v_1, ..., v_{i-1}, v, v_i, ..., v_k)$$

is $F$-linear in $v$. Then the universal mapping property of $T(V)$ can be interpreted as follows: If $\lambda$ is a $k$-multilinear map of $V \times V \times \cdots \times V$ to $W$ then there exists a unique linear map $\widetilde{\lambda} : \bigotimes^k V \to W$ such that $\lambda(v_1, ..., v_k) = \widetilde{\lambda}(v_1 \bigotimes v_2 \bigotimes \cdots \bigotimes v_k)$.

We say that a $k$-multilinear map $\lambda : V \times V \times \cdots \times V \to W$ is alternating if $\lambda(v_1, ..., v_k) = 0$ whenever $v_i = v_{i+1}$ for some $1 \leq i < k$. This implies that

$$0 = \lambda(v_1, ..., v_i + v_{i+1}, v_i + v_{i+1}, ..., v_k) =$$

$$\lambda(v_1, ..., v_i, v_i, ..., v_k) + \lambda(v_1, ..., v_i, v_{i+1}, ..., v_k) +$$

$$\lambda(v_1, ..., v_{i+1}, v_i, ..., v_k) + \lambda(v_1, ..., v_{i+1}, v_{i+1}, ..., v_k) =$$

$$\lambda(v_1, ..., v_i, v_{i+1}, ..., v_k) + \lambda(v_1, ..., v_{i+1}, v_i, ..., v_k).$$

Thus, if $s \in S_k$ (the group of permutations in $k$ letters) then

$$\lambda(v_{s1}, ..., v_{sk}) = sgn(s)\lambda(v_1, ..., v_k).$$

Here *sgn* is the usual sign of a permutation. This is a consequence of the fact that $S_n$ is generated by the transpositions $(i, i+1)$. We see that $\bigwedge^k V$ has the universal mapping property:If $\lambda$ is an alternating $k$-multilinear map of $V \times V \times \cdots \times V$ to $W$ then there exists a unique linear map $\widetilde{\lambda} : \bigwedge^k V \to W$ such that $\lambda(v_1, ..., v_k) = \widetilde{\lambda}(v_1 \bigwedge v_2 \bigwedge \cdots \bigwedge v_k)$.

**Lemma 1** *If $v_i$, $i \in I$ (a totally ordered set) is a basis of $V$ over $F$ then the elements $v_{i_1} \bigwedge v_{i_2} \bigwedge \cdots \bigwedge v_{i_k}$ for $i_1 < ... < i_k$ form a basis of $\bigwedge^k V$.*

**Proof.** Let $\lambda_i, i \in I$ be the dual elements to $v_i, i \in I$ (that is, $\lambda_i(v_j) = \delta_{ij}$). Define for $i_1 < ... < i_k$ the element

$$\lambda_{i_1 \ldots i_k}(w_1, ..., w_k) = \det\left[\lambda_{i_j}(w_l)\right]_{1 \leq j,l \leq k}.$$

The universal mapping property implies that there exist linear maps $\widetilde{\lambda_{i_1 \ldots i_k}} : \bigwedge^k V \to W$ such that if $i_1 < ... < i_k$ and $l_1 < ... < l_k$

$$\widetilde{\lambda_{i_1 \ldots i_k}}(v_{l_1} \bigwedge v_{l_2} \bigwedge \cdots \bigwedge v_{l_k}) = \prod_{m=1}^{k} \delta_{i_m l_m}.$$

Evaluating $\widetilde{\lambda_{l_1 \ldots l_k}}$ with $l_1 < ... < l_k$ on an identity

$$\sum_{i_1 < \ldots < i_k} a_{i_1 \cdots i_k} v_{i_1} \bigwedge v_{i_2} \bigwedge \cdots \bigwedge v_{i_k} = 0$$

yields $a_{l_1 \cdots l_k} = 0$. This the elements are linearly independent. Since they span they must be a basis. ■

**Corollary 2** *If* $\dim V = n < \infty$ *then* $\dim \bigwedge^k V = \binom{n}{k}$. *In particular,* $\dim \bigwedge V = 2^n$.

We note that if $W \subset V$ is a subspace then the injection $i : W \to V \subset \bigwedge V (i(w) = w)$ induces an algebra homomorphism of $\bigwedge W$ into $\bigwedge V$. The previous lemma implies that the map is injective. Thus we can think of $\bigwedge W$ as a subalgebra of $\bigwedge V$.

**Lemma 3** *Let $W$ be a $k$-dimensional subspace of $V$. Set $u = \bigwedge^k W$ then $v \in V$ is an element of $W$ if and only if $v \bigwedge u = 0$.*

**Proof.** The necessity is obvious since $\bigwedge^{k+1} W = 0$. Let $w_1, ..., w_k$ be a basis of $W$ extend to an ordered basis $\{w_1, ..., w_k\} \cup \{\ v_i | i \in I\}$ of $V$. If $v \in V$ then $v = \sum_{i=1}^k a_i w_i + \sum b_j v_j$ a finite sum. Now $Fu = \bigwedge^k W = Fw_1 \bigwedge w_2 \bigwedge \cdots \bigwedge w_k$ so

$$v \bigwedge u = (-1)^k \sum b_j w_1 \bigwedge w_2 \bigwedge \cdots \bigwedge w_k \bigwedge v_j.$$

Since the previous lemma implies that the elements $w_1 \bigwedge w_2 \bigwedge \cdots \bigwedge w_k \bigwedge v_j$ are linearly independent the lemma follows. ■

The next result is useful in several contexts.

**Lemma 4** *Let* $v \neq 0$, $v \in V$ *and let* $\varepsilon(v) : \bigwedge V \to \bigwedge V$ *be given by* $\varepsilon(v)u = v \bigwedge u$. *Then the sequence*

$$0 \to F \xrightarrow{\varepsilon(v)} V \xrightarrow{\varepsilon(v)} \bigwedge^2 V \to \dots \xrightarrow{\varepsilon(v)} \bigwedge^k V \xrightarrow{\varepsilon(v)} \dots$$

*is exact.*

**Proof.** We can think of $v$ as the first element of a basis of $V$ then Lemma 1 implies the assertion. ■

If $\lambda \in V^*$(the space of linear maps of $V$ to $F$) then we define

$$\iota(\lambda)(v_1, ..., v_k) = \sum (-1)^{i+1} \lambda(v_i) v_1 \bigwedge v_2 \bigwedge \cdots \bigwedge \widehat{v_i} \bigwedge \cdots \bigwedge v_k$$

then this defines an alternating $k$-multilinear map to $\bigwedge^{k-1} V$. Hence induces a linear map $\iota(\lambda) : \bigwedge^k V \to \bigwedge^{k-1} V$.

**Lemma 5** *An element* $u \in \bigwedge^k V$ *is of the form* $u = v_1 \bigwedge v_2 \bigwedge \cdots \bigwedge v_k$ *with* $v_i \in V$ *if and only if for every choice of* $\lambda_1, ..., \lambda_{k-1} \in V^*$

$$(\iota(\lambda_1) \cdots \iota(\lambda_{k-1})u) \bigwedge u = 0.$$

**Proof.** Set $L(u) = Span_F\{(\iota(\lambda_1) \cdots \iota(\lambda_{k-1})u) | \lambda_1, ..., \lambda_{k-1} \in V^*\}$. If $u = v_1 \bigwedge v_2 \bigwedge \cdots \bigwedge v_k$ and $u \neq 0$ then $L(u) = Span_F\{v_i | i = 1, .., k\}$. So the necessity follows from Lemma 3. Suppose that the condition is satisfied. We assume that $u \neq 0$. Then it is easy to see that $\dim L(u) \geq k$ and that $u \in \bigwedge^k L(u)$. The previous lemma implies that if $v \in L(u)$, $v \neq 0$ then since $v \bigwedge u = 0$ we have $u = v \bigwedge u'$ for some $u' \in \bigwedge^{k-1} L(u)$. Let $v_1, v_2, ...$ be a basis of $L(u)$. Then $u = v_1 \bigwedge u_1$ with $u_1 \in \bigwedge^{k-1} Span_F\{v_2, ...\}$. Thus since

$$0 = v_2 \bigwedge u = v_2 \bigwedge v_1 \bigwedge u_1 = -v_1 \bigwedge v_2 \bigwedge u_1$$

Lemma 1 implies that $v_2 \bigwedge u_1 = 0$. So $u_1 = v_2 \bigwedge u_2$ with $u_2 \in \bigwedge^{k-2} Span_F\{v_3, ...\}$. Continuing in this way $k$ steps we find that $u = v_1 \bigwedge v_2 \bigwedge \cdots \bigwedge v_k \bigwedge u_k$ with $u_k \in \bigwedge^0 Span_F\{v_{k+1}, ...\} = F$. ■

## 2 The Grassmann varieties.

Let $V$ be an $n$-dimensional vector space over $F$. We set $Gr_k(V) = \{W | W$ a $k$-dimensional subspace of $V\}$. If $V = F^n$ then we write $Gr_{k,n} = Gr_k(V)$. Obviously, a choice of basis induces a bijection between $Gr_k(V)$ and $Gr_{k,n}$ if $\dim V = n$. If $W \in Gr_{k,n}$ then we have a linear isomorphism $x : F^k \to W \subset F^n$. We can think of $x$ as an $n \times k$ matrix. Since $\dim W = k$ we see that $rank(x) = k$. Conversely if $x$ is an $n \times k$ matrix of rank $k$ then $\dim xF^k = k$. Let for $1 \leq i_1 < i_2 < ... < i_k \leq n$, $X_{i_1 \dots i_k}$ be the space of $n \times k$ matrices such that the

$i_j$ row is $[0,0,...,0,1,0,...,0]$ with the 1 in the $j$-th position. Thus $X_{12...k}$ is the set of all matrices of the form

$$\begin{bmatrix} I \\ z \end{bmatrix}$$

with $I$ the $k \times k$ identity matrix and $z$ is an arbitrary $n-k \times k$ matrix.

**Lemma 6** *Let $1 \le i_1 < i_2 < ... < i_k \le n$. The map $\Phi_{i_1...i_k} : X_{i_1...i_k} \to Gr_{k,n}$ given by $\Phi_{i_1...i_k}(x) = xF^k$ is injective. Furthermore,*

$$Gr_{k,n} = \bigcup_{i_1<...<i_k} \Phi_{i_1...i_k}(X_{i_1...i_k}).$$

**Proof.** Let $e_1,...,e_n$ be the standard basis of $F^n$. We will prove the first assertion for the case when $i_j = j$, for $j = 1,...,k$. The general case will follow by observing that if $\sigma$ is a permutation in $S_n$ acting on $F^n$ by permuting the coordinates then if $sj = i_j$ then $s\Phi_{12...k}(s^{-1}x) = \Phi_{i_1...i_k}(x)$. Set $\Phi = \Phi_{1...k}$., $X = X_{1...k}$ Then if

$$\Phi \begin{bmatrix} I \\ z \end{bmatrix} = \Phi \begin{bmatrix} I \\ w \end{bmatrix}$$

then we must have for each $j = 1,...,k$ the relation $e_j + \sum_{i>k} w_{ji}e_i = \sum_{l=1}^{k} bj_l(e_l + \sum_{i>k} z_{li}e_i)$. This implies that $b_{jl} = \delta_{jl}$ and $w_{ji} = z_{ji}$.

As for the second assertion we note that if $W \in Gr_{k,n}$ then $W = xF^k$ with $x$ an $n \times k$ matrix of rank $k$. This implies that there exist $1 \le i_1 < ... < i_k \le n$ such that if $x_j$ is the $j$-th row of $x$ then the matrix

$$z = \begin{bmatrix} x_{i_1} \\ x_{i_2} \\ \vdots \\ x_{i_k} \end{bmatrix}$$

is invertible. We then have $xz^{-1} \in X_{i_1...i_k}$. Clearly $xz^{-1}F^k = xF^k$. ■

We define for $1 \le i_1 < ... < i_k \le n$, $T_{i_1 \cdots i_k} : M_{n-k,k}(F) \to X_{i_1 \cdots i_k}$ by putting the element $e_1$ in as the $i_1$ row and shifting everyting up one unit. Then putting $e_2$ in as the $i_2$ row and shifting,... Then if we set $U_{i_1 \cdots i_k} = \Phi_{i_1...i_k}(X_{i_1...i_k})$ and if we write $\Psi_{i_1 \cdots i_k} = \Phi_{i_1...i_k} T_{i_1 \cdots i_k}$ then

$$\Psi_{i_1 \cdots i_k} : M_{n-k,k}(F) \to U_{i_1 \cdots i_k}$$

is bijective. We note that $M_{n-k,k}(F)$ is an $(n-k)k$ dimensional vector space over $F$. Thus we can endow it with the Zariski topology and consider it to be $\mathbb{A}^{k(n-k)}$.

**Lemma 7** *The set $\Psi^{-1}_{i_1 \cdots i_k}(U_{i_1 \cdots i_k} \cap U_{j_1 \cdots j_k})$ is open in $M_{n-k,k}(F)$.*

**Proof.** This set can be desctibed as follows. Take $x \in M_{n-k,k}(F)$. Then inset the $e_i$, $i = 1,..,k$ in the indicated positions to get an element of $X_{i_1 \cdots i_k}$ and let $f$ be the minor of this matrix which is the determinant of the matri consisting of the rows $j_1, ..., j_k$. Then $f$ is a polynomial on $M_{n-k,k}(F)$ and the indicated set is the set of all elements $y$ with $f(y) \neq 0$. $\blacksquare$

This lemma allows us to define a topology on $Gr_{k,n}$ by taking the open sets to be unions of images under the $\Psi_{i_1 \cdots i_k}$ of open sets in $M_{n-k,k}(F)$. If $U \subset Gr_{k,n}$ is open then we set $\mathcal{O}_{Gr_{k,n}}(U)$ equal to the set of all functions such that for each $i_1 < ... < i_k$ then

$$f \circ \Psi_{i_1 \cdots i_k | \Psi^{-1}_{i_1 \cdots i_k}(U \cap U_{i_1 \cdots i_k})} \in \mathcal{O}_{M_{n-k,k}(F)}(\Psi^{-1}_{i_1 \cdots i_k}(U \cap U_{i_1 \cdots i_k}).$$

One checks that with this definition $(Gr_{k,n}, \mathcal{O}_{Gr_{k,n}})$ is a space with functions (in the sense of Kempf). Since $(U_{i_1 \cdots i_k}, \mathcal{O}_{Gr_{k,n}|U_{i_1 \cdots i_k}})$ is isomorphic with $\mathbb{A}^{(n-k)k}$ as a space with functions we see that with this struction $Gr_{k,n}$ is a variety (in the sense of Kempf).

Our next task is to show that it is projective. We note that if $k = 1$ then $G_{k,n} = \mathbb{P}^{n-1}$(as a set). Furthermore, the spaces $U_i$ $i = 1, ..., n$ in this case are just the spaces $\mathbb{P}^{n-1}_{i-1}$ in our discussion of projective space. Thus the equation is also as a variety). We will thus consider $Gr_1(V)$ to be isomorphic as a variety with $\mathbb{P}^{n-1}$ for $V$ an $n$-dimensional space of $F$

We set $V = F^n$ and define a map $\Gamma : Gr_{k,n} \to \mathbb{P}(\bigwedge^k V) = \mathbb{P}^{\binom{n}{k}-1}$ (this involves choosing bases) by $\Gamma(W) = \bigwedge^k W$ (note that this is a one dimensional subspace of $\bigwedge^k V$. We now have the main result.

**Theorem 8** *We have* $\Gamma(Gr_{k,n})$ *is closed in* $\mathbb{P}(\bigwedge^k V)$ *and the map* $\Gamma$ *defines an isomorphism of* $Gr_{k,n}$ *with the projective variety* $\Gamma(Gr_{k,n})$.

**Proof.** Lemma 3 implies that $\Gamma$ is one to one. In the notation of Lemma 5 we define

$$f_{\lambda_1, ..., \lambda_{k-1}; \mu}(u) = \mu((\iota(\lambda_1) \cdots \iota(\lambda_{k-1})u) \bigwedge u)$$

for $\lambda_1, ..., \lambda_{k-1} \in V^*, \mu \in \left(\bigwedge^{k+1} V\right)^*$ and $u \in \bigwedge^k V$. Then $f_{\lambda_1, ..., \lambda_{k-1}; \mu}(u)$ is a homogenious polynomial of degree 2 in $u$ and the set of joint zeros of the $f_{\lambda_1, ..., \lambda_{k-1}; \mu}$ is the image of $\Gamma$. thus the image is closed. We will now show that $\Gamma_{|U_{i_1 \cdots i_k}}$ is an isomorphism onto its image for each $i_1 < ... < i_k$. Up to permuting the coordinates of $F^n$ it is enought to prove this for $i_j = j$. In this case we consider $x \in M_{n-k,k}$ to be $k$ column vectors in $F^n$ that are linear combinations of $e_{k+1}, ..., e_n$. Then we have

$$\Gamma(\Psi_{1 \ldots k}(x)) = [(e_1 + x_1) \bigwedge (e_2 + x_2) \bigwedge \cdots \bigwedge (e_k + x_k)]$$

(here, as usual, $[z]$ is the line containing the nonzero vector $z$). This implies that the map is regular and we have

$$(e_1 + x_1) \bigwedge (e_2 + x_2) \bigwedge \cdots \bigwedge (e_k + x_k) = e_1 \bigwedge e_2 \bigwedge \cdots \bigwedge e_k +$$

$$\sum_{i=1}^{k} (-1)^{i+1} x_i \bigwedge e_1 \bigwedge e_2 \bigwedge \cdots \bigwedge \widehat{e_i} \bigwedge \cdots \bigwedge e_k + ....$$

If we choose as a basis of $\bigwedge^k V$ the elements $e_{i_1} \bigwedge e_{i_2} \bigwedge \cdots \bigwedge e_{i_k}$ with $i_1 < ... < i_k$ and we order the indices as $0, 1, ..., \binom{n}{k} - 1$ with the 0-th basis element being $e_1 \bigwedge e_2 \bigwedge \cdots \bigwedge e_k$ then the formula implies that $\Gamma(U_{1 \cdots k}) \subset \mathbb{P}_0^{\binom{n}{k}-1}$ and the map is an ismorphism onto its image. ■