1 The groups and Lie algebras

We consider $G \subset GL(n, \mathbb{C})$ (n × n invertible matrices) a subgroup given as the locus of zeros of polynomial in $\mathbb{C}[x_{ij}]$ with $x_{ij}$ the matrix entries of an $n \times n$-matrix that is an element of $M_n(\mathbb{C})$. $G$ in the jargon of algebraic group theory is an affine algebraic group, i.e. a Zariski closed subgroup of $GL(n, \mathbb{C})$. Set $X^T$ equal to the transpose of $X$ and $X^* = X^T$ as usual.

Let $\text{Lie}(G)$ denote its Lie algebra which we can take to be all $X \in M_n(\mathbb{C})$ such that $e^{tX} \in G$ for all $t \in \mathbb{C}$. Here if $\|X\| = \text{tr}(XX^*)^{1/2}$ then $\|\cdots\|$ defines a norm on the vector space $M_n(\mathbb{C})$ and

$$\|XY\| \leq \|X\| \|Y\|, X,Y \in M_n(\mathbb{C}).$$

Thus the series

$$e^X = \sum_{m=0}^{\infty} \frac{X^m}{m!}$$

converges uniformly and absolutely on closed balls $\|X\| \leq r$. This implies that $X \mapsto e^X$ defines a complex analytic function from $M_n(\mathbb{C})$ to the open subset $GL(n, \mathbb{C})$. It is standard that if $X,Y \in \text{Lie}(G)$ then

$$[X,Y] = XY - YX \in \text{Lie}(G).$$

As usual, we set $U(n) = \{g \in M_n(\mathbb{C})|g^*g = I\}$. Then $U(n)$ is a compact subgroup of $GL(n, \mathbb{C})$ (it is a closed subset of the unit sphere in $M_n(\mathbb{C})$) and so is a Lie subgroup. Its Lie algebra is precisely the set of skew-hermitian $n \times n$ matrices.

If we assume that if $g \in G$ then $g^* \in G$, that is, $G$ is a symmetric subgroup. We set $K = G \cap U(n)$. Then $K$ is also a compact Lie group and $\text{Lie}(K)$ is the set of all elements in $\text{Lie}(G)$ that are skew-Hermitian.

**Theorem 1** (cf. [GIT, Theorem 37]) Let $H$ be a Zariski closed, reductive subgroup of $G$, a symmetric subgroup of $GL(n, \mathbb{C})$ then there exists $g \in G$ such that $gHg^{-1}$ is symmetric.

Thus we can take as a definition of reductive that there exists an imbedding in $GL(n, \mathbb{C})$ as a symmetric subgroup.

We assume that $G$ is symmetric. A basic structural result is

$$\text{Lie}(G) = \text{Lie}(K) \oplus i\text{Lie}(K)$$
as a real vector space and the map

\[ K \times i\text{Lie}(K) \to G \]

\[ k, X \mapsto ke^X \]

is a diffeomorphism onto \( G \). Clearly this implies that \( K \) is a maximal compact subgroup of \( G \). One can prove that all maximal compact subgroups of \( G \) are conjugate to \( K \). In particular, any compact subgroup of \( G \) can be conjugated into \( K \).

As usual we denote by \( Ad(g)X \) the action of the inner automorphism of \( \text{Lie}(G) \) corresponding to \( g \in G \) on \( X \). That is, \( Ad(g)X = gXg^{-1} \). We also write \( \text{Inn}(g)x = gxg^{-1} \) for \( g, x \in G \).

On \( \text{Lie}(G) \) we put the bilinear form \( (X, Y) = \text{tr}XY \). We note that \((...,...)\) is \( G \)-invariant, that is,

\[ (gXg^{-1}, gXg^{-1}) = \text{tr}(gXYg^{-1}) = \text{tr}(XY) = (X, Y). \]

Then since \( G \) is invariant under \( g \mapsto g^* \) (hence so is \( \text{Lie}(G) \)) the form is nondegenerate.

**Lemma 2** If \( H \subset G \) is Zariski closed and reductive then \((...,...)|_{\text{Lie}(H)}\) is non-degenerate.

**Proof.** We have observed above that there exists \( g \in G \) so that \( gHg^{-1} \) is invariant under \( h \mapsto h^* \). Hence the form is nondegenerate on \( g\text{Lie}(H)g^{-1} \). The \( G \)-invariance of the form implies the result. \( \blacksquare \)

We note that the converse is also true. Here are examples of reductive groups

1. \( GL(n, \mathbb{C}) \).
2. \( SL(n, \mathbb{C}) = \{ g \in GL(n, \mathbb{C}) | \text{det}(g) = 1 \} \).
3. \( O(n, \mathbb{C}) = \{ g \in GL(n, \mathbb{C}) | gIg^T = I \} \).
4. \( SO(n, \mathbb{C}) = O(n, \mathbb{C}) \cap SL(n, \mathbb{C}) \).
5. Let

\[ J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \]

with \( I \) the \( n \times n \) identity matrix. Set \( Sp(n, \mathbb{C}) = \{ g \in GL(n, \mathbb{C}) | gJg^T = J \} \).

**Exercise.** These five groups and arbitrary finite products of them are symmetric subgroups.

These groups and their coverings and quotients give the full set of classical groups.
2 Gradings of Lie algebras

Let \( \mathfrak{g} \) be a Lie algebra over \( \mathbb{C} \) and let \( S \) be a commutative semi-group (written additively) then an \( S \)-grading of \( \mathfrak{g} \) is a direct sum decomposition

\[
\mathfrak{g} = \bigoplus_{s \in S} \mathfrak{g}_s
\]

with \( \mathfrak{g}_s \) possibly 0 and

\[
[\mathfrak{g}_s, \mathfrak{g}_t] \subset \mathfrak{g}_{s+t}
\]

for \( s, t \in S \).

2.1 \( \mathbb{Z} \)-gradings

The main examples that are of interest to us are when \( S = \mathbb{Z} \) or \( S = \mathbb{Z}/n\mathbb{Z} \) with \( n \) a strictly positive integer. We will only be looking at finite dimensional Lie algebras.

We have the following characterization of a \( \mathbb{Z} \)-grading.

**Lemma 3** A \( \mathbb{Z} \)-grading of \( \mathfrak{g} \) is equivalent to having an algebraic homomorphism \( \phi : \mathbb{C}^\times \to \text{Aut}(\mathfrak{g}) \) via \( \phi(z)x = z^n x \) for \( x \in \mathfrak{g}_n \).

**Proof.** We leave it to the reader to check that the formula in the statement defines an algebraic homomorphism of \( \mathbb{C}^\times \) to \( \text{Aut}(\mathfrak{g}) \). If \( \phi \) is an algebraic homomorphism of \( \mathbb{C}^\times \) to \( \text{Aut}(\mathfrak{g}) \) then \( \mathfrak{g} \) is the direct sum of spaces \( \mathfrak{g}_\chi \) with \( \phi(z)x = \chi(z)x \) for \( x \in \mathfrak{g}_\chi \) with \( \chi \) an algebraic homomorphism of \( \mathbb{C}^\times \) to itself (i.e. an algebraic character). The algebraic characters of \( \mathbb{C}^\times \) are exactly the ones given by \( \chi_n(z) = z^n \). If we set \( \mathfrak{g}_n = \mathfrak{g}_{\chi_n} \) we have our grade. ■

**Exercises.**

1. When is a \( \mathbb{Z}/n\mathbb{Z} \) grading a \( \mathbb{Z} \)-grading?

2. Let \( k > 0 \) be an integer and let \( \mathfrak{g} = \bigoplus_{s \in \mathbb{Z}} \mathfrak{g}_s \) be a \( \mathbb{Z} \)-grading of \( \mathfrak{g} \). Let for \( n \in \mathbb{Z} \), \([n]\) be its class mod \( k \). Then defining \( \mathfrak{g}_{[n]} = \bigoplus_{r \equiv n \text{ mod } k} \mathfrak{g}_r \) we have a \( \mathbb{Z}/n\mathbb{Z} \) grading.

2.2 \( \mathbb{Z}/n\mathbb{Z} \) gradings

We now consider the case when we have a \( \mathbb{Z}/n\mathbb{Z} \)-grading. Then if \( \zeta \) is an \( n \)-th root of unity we can define \( \theta : \mathfrak{g} \to \mathfrak{g} \) by \( \theta_{[n]} = \zeta^n I \). Then \( \theta \) defines an automorphism of \( \mathfrak{g} \) of order \( n \). If \( \theta \) is an automorphism of order \( n \) and if \( \zeta \) is a primitive \( n \)-th root of unity then we can define \( \mathfrak{g}_{[n]} = \{ x \in \mathfrak{g} | \theta x = \zeta^n x \} \) and this is a \( \mathbb{Z}/n\mathbb{Z} \) grading.
2.3 Relationship with groups 1. \( \mathbb{Z} \)-grades.

We now consider the case when \( G \) is a connected reductive algebraic group over \( \mathbb{C} \) and \( \mathfrak{g} = \text{Lie}(G) \) and \( \mathfrak{g} \) is \( \mathbb{Z} \)-graded

\[
\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_j
\]

then let \( \phi : \mathbb{C}^* \to \text{Aut}(\mathfrak{g}) \) be the algebraic homomorphism

\[
\phi(z)v = z^j v, v \in \mathfrak{g}_j.
\]

Let \( \mathfrak{g} = \mathfrak{c} \oplus [\mathfrak{g}, \mathfrak{g}] \) with \( \mathfrak{c} \) the center of \( \mathfrak{g} \) and \([\mathfrak{g}, \mathfrak{g}]\) the commutator algebra which is semi-simple. We observe that

\[
\phi(\mathbb{C}^*)\mathfrak{c} = \mathfrak{c}
\]

and

\[
\phi(\mathbb{C}^*)[\mathfrak{g}, \mathfrak{g}] \subseteq [\mathfrak{g}, \mathfrak{g}].
\]

Let \( C \) be the center of \( G \) and let \( G' \) be the derived group then \( G = CG' \). \( G' \) is connected and \( \text{Lie}(G') = [\mathfrak{g}, \mathfrak{g}] \) inherits a grade from \( \mathfrak{g} \). \( \text{Ad}(G) = \text{Ad}(G') \) and since all of the group actions that Vinberg studies are by subgroups of the adjoint group of the Lie algebra we will emphasize \( G' \) and \( \mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}] \). For the rest of this subsection we will assume that \( \mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \). The linear map \( D : \mathfrak{g} \to \mathfrak{g} \) given by

\[
D_{[\mathfrak{g}, \mathfrak{g}]} = jI
\]

defines a derivation of \( \mathfrak{g} \). All derivations of \( \mathfrak{g} \) are inner so \( D = \text{ad}(x) \) with \( x \in \mathfrak{g} \). We have proved

**Lemma 4** If \( G \) is a connected semisimple algebraic group over \( \mathbb{C} \) with a \( \mathbb{Z} \)-graded Lie algebra then there exists a semisimple element (i.e. diagonalizable) \( x \in \text{Lie}(G) \) such that \( \text{ad}(x) \) has integral eigenvalues and the eigenspaces of \( \text{ad}(x) \) give the grade.

2.4 Relationship with groups 2. \( \mathbb{Z}/n\mathbb{Z} \)-grades.

We now assume that \( G \) is a connected reductive algebraic group with \( \mathfrak{g} = \text{Lie}(G) \) having a \( \mathbb{Z}/n\mathbb{Z} \)-grade,

\[
\mathfrak{g} = \bigoplus_{[j] \in \mathbb{Z}/n\mathbb{Z}} \mathfrak{g}_{[j]}.
\]
Let \( \zeta \) be a primitive \( n \)-th root of unity and

\[ \theta|_{\mathfrak{g}_[j]} = \zeta^j I. \]

As above

\[ \mathfrak{g} = \mathfrak{c} \oplus [\mathfrak{g}, \mathfrak{g}] \]

and \( \theta \mathfrak{c} = \mathfrak{c} \) and \( \theta[\mathfrak{g}, \mathfrak{g}] = [\mathfrak{g}, \mathfrak{g}] \). The same non-interaction between the grades of the center and the derived algebras prevails.

Assuming that \( G \) is semi-simple (i.e. \( \mathfrak{c} = 0 \)) then \( \text{Ad} : G \to \text{Ad}(G) \) is a covering homomorphism and \( \text{Ad}(G) \) is the identity component of \( \text{Aut}(G) \).

We have

**Lemma 5** Let \( G \) be a connected semi-simple Lie group with a \( \mathbb{Z}/n\mathbb{Z} \)-graded Lie algebra \( \mathfrak{g} = \text{Lie}(G) \). If \( \theta \) is as above and \( \tau(g) = \theta g \theta^{-1} \) for \( g \in \text{Ad}(G) \) then identifying \( \mathfrak{g} \) with \( \text{ad}(\mathfrak{g}) \), \( d\tau = \theta \).

### 2.5 The affine action associated with a grade.

Let \( G \) be a connected reductive algebraic group with \( \mathfrak{g} = \text{Lie}(G) \) having a \( \mathbb{Z} \) or a \( \mathbb{Z}/m\mathbb{Z} \) grade with \( m \in \mathbb{Z}_{>0} \). So we have a graded decomposition

\[ \mathfrak{g} = \bigoplus_{s \in S} \mathfrak{g}_s \]

with \( S = \mathbb{Z} \) or \( \mathbb{Z}/m\mathbb{Z} \). In both cases we will write \([j]\) for the corresponding element of \( S \). Let \( \mathfrak{h} \) denote the Lie algebra \( \mathfrak{g}_{[j]} \) and let \( V = \mathfrak{g}_{[1]} \). Let \( H \) be the connected Lie subgroup of \( G \) with Lie algebra \( \mathfrak{h} \). Then the pair \( (\text{Ad}(H)|_V, V) \) of a group and module for the group is what we will call the affine action associated with the grade.

**Lemma 6** The group \( \text{Ad}(H)|_V \) is a connected reductive algebraic subgroup of \( GL(V) \) that depends on \( \mathfrak{g} \) and not the choice of \( G \).

**Proof.** We note that \( V = \mathfrak{c} \cap \mathfrak{g}_{[1]} \oplus [\mathfrak{g}, \mathfrak{g}] \cap \mathfrak{g}_{[1]} \). Relative to this decomposition the elements of \( H \) act trivially on the first summand and preserve the second. We may therefore assume that \( \mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \). If \( S = \mathbb{Z} \) let \( x \in \mathfrak{g} \) be such that \( dx \) gives the grade. If \( S = \mathbb{Z}/m\mathbb{Z} \) then let \( \theta \) be as in the previous subsection. In the first case we note that \( H \) is the identity component of \( \{ h \in H | \text{Ad}(h)x = x \} \) and in the second \( H \) is the identity component of
\{h \in H | Ad(h)\theta = \theta Ad(h)\}$ in both cases this defines $Z$–closed subgroup of $G$ and since $x$ and $\theta$ are semi-simple $H$ is reductive. In both cases $Ad(H)|_V$ is the connected subgroup of $GL(V)$ with lie algebra $\{ady | y \in \mathfrak{h}\}$ so it depends only on $\mathfrak{g}$. ■

This leads us to the main objects of our study. We use the term $Z$-closed for Zariski closed.

**Definition 7** A Vinberg pair will be a pair $(L, V)$ with $L$ a connected, reductive $Z$-closed subgroup of $GL(V)$ such that there exists a graded reductive Lie algebra

$$\mathfrak{g} = \bigoplus_{s \in S} \mathfrak{g}_s$$

with $S = \mathbb{Z}$ or $\mathbb{Z}/m\mathbb{Z}$ such that $V = \mathfrak{g}[1]$ and $L = Ad(H)|_V$ with $H$ as in the previous Lemma.

**Definition 8** A direct sum of Vinberg pairs is a pair $(L, V)$ with $L = L_1 \times \cdots \times L_r$ and $V = V_1 \oplus V_2 \oplus \cdots \oplus V_r$ with $(L_i, V_i)$ a Vinberg pair and $L$ acts by the block diagonal action.

**Definition 9** A morphism of direct sums of Vinberg pairs $(L, V)$ and $(L', V')$ is a pair, $(\phi, T)$ of an algebraic group homomorphism $\phi : L \to L'$ and a linear map $T : V \to V'$ such that $Tgv = \phi(g)Tv$ for $g \in L$ and $v \in V$.

**Exercise.** When is a direct sum of Vinberg pairs isomorphic with a Vinberg pair?

### 2.6 Decomposition into simple constituents.

In the last section we showed how one can assign a reductive algebraic group, $H$, and an algebraic $H$–module $V$ to a reductive graded Lie algebra. In this section we will describe a decomposition of into what we will call simple constituents.

Let $G$ and $\mathfrak{g}$ be as in the previous section. Let

$$\mathfrak{g} = \mathfrak{c} \oplus \bigoplus_{j=1}^{n} \mathfrak{g}^{(j)}$$

be a decomposition of $\mathfrak{g}$ into the direct sum of its center, $\mathfrak{c}$, and its simple ideals

$$\mathfrak{g}^{(1)}, \mathfrak{g}^{(2)}, \ldots, \mathfrak{g}^{(n)}.$$
If the grade on \( g \) is a \( \mathbb{Z} \)-grade then we have seen that each of the summands inherits the grade. So in this case a \( \mathbb{Z} \)-graded simple constituent will just be a \( \mathbb{Z} \)-graded simple Lie algebra. We have

**Lemma 10** Notation as above. If \( g \) is \( \mathbb{Z} \)-graded then \( g = c \oplus \bigoplus_{j=1}^{n} g^{(j)} \) is a decomposition of \( g \) into a direct sum of \( \mathbb{Z} \)-graded Lie algebras. Furthermore the corresponding affine group action is given as follows

\[
V = c[1] \oplus g^{(1)}[1] \oplus \ldots \oplus g^{(n)}[1] = c[1] \oplus V^{(1)} \oplus \ldots \oplus V^{(n)}
\]

and if we take \( G^{(i)} \) to be the connected subgroup of \( G \) with Lie algebra \( g^{(i)} \) and \( H^{(i)} \) the connected subgroup of \( G^{(i)} \) with Lie algebra \( g^{(i)}[0] \) then

\[
\text{Ad}(H)|_V = \{I_{c[1]} \} \times \text{Ad}(H^{(1)}|_{V^{(1)}}) \times \cdots \times \text{Ad}(H^{(n)}|_{V^{(n)}}).
\]

We now consider the case when the grade is a \( \mathbb{Z}/m \mathbb{Z} \)-grade. The situation is almost the same except for one complication.

In this case we define \( \theta : g \to g \) by \( \theta|_{g^{(j)}} = \zeta^j I \) with \( \zeta \) a primitive \( m \)-the root of unity. Then \( \theta c = c \) and \( \theta \) permutes the simple factors. In this case the permutation breaks up into disjoint cycles. We consider one such cycle of length \( d \) and relabel so that the cycle is

\[
1 \to 2 \to \ldots \to d \to 1.
\]

We note that \( \theta \) preserves \( \tilde{g} = \bigoplus_{j=1}^{d} g^{(j)} \) and we analyze its action on this space. If \( x \in \bigoplus_{j=1}^{d} g^{(j)} \) then \( x = \sum x_i \) and relative to this decomposition we have

\[
\theta x = \sum \theta x_i
\]

And \( \theta : g^{(j)} \to g^{(j+1)} \) for \( j = 1, \ldots, d-1 \) and \( \theta : g^{(d)} \to g^{(1)} \) is a Lie algebra isomorphism. We denote the \( \eta \)-eigenspace for \( \theta \) by sub-\( \eta \). Thus \( \tilde{g}_1 = g^{[0]} \) consists of the elements such that \( \theta x_i = x_{i+1} \) for \( i = 1, \ldots, d-1 \) and \( \theta x_d = x_1 \).

So \( x \) can be written

\[
\sum_{i=0}^{d-1} \theta^i x_1
\]

with the additional condition \( \theta^d x_1 = x_1 \). Noting that \( [g^{(i)}, g^{(j)}] = 0 \) if \( i \neq j \) we see that if we set \( \theta^{(1)} \) equal to \( \theta^d \) then the map \( \phi_1 : g^{(1)}_1 \to \tilde{g}_1 \) given by

\[
\phi_1(x_1) = \sum_{i=0}^{d-1} \theta^i x_1
\]

\[
7
\]
is a Lie algebra isomorphism. Furthermore if \( \eta \) is an \( m \)-th root of unity and if \( x \in \tilde{\mathfrak{g}} \eta \) then we have \( \theta x_i = \eta x_{i+1}, i = 1, ..., d - 1 \) and \( \theta x_d = \eta x_1 \) so \( x_i = \eta^{-i} \theta^i x_1 \) and as before \( \theta^d x_1 = \eta^d x_1 \). We see that the map \( \phi_\eta : \mathfrak{g}^{(1)}_{\eta^d} \to \tilde{\mathfrak{g}}_\eta \) given by

\[
\phi_\eta(x_1) = \sum_{i=0}^{d-1} \eta^{-i} \theta^i x_1
\]

defines a linear isomorphism that it addition satisfies

\[
\phi_\eta([x_1, y_1]) = [\phi_\eta(x_1), \phi_\eta(y_1)]
\]

for \( x_1 \in \mathfrak{g}^{(1)} \) and \( y_1 \in \mathfrak{g}^{(1)}_{\eta^d} \).

We can now prove the analogue of Lemma 10.

**Proposition 11** Let \((L, V)\) be a Vinberg pair. Then \((L, V)\) is isomorphic with a direct sum of a vector space \((\{I\}, V_0)\) and Vinberg spaces \((L_j, V_j)\) with corresponding graded Lie algebra simple.

The above argument suggests the following let \( \mathfrak{g} \) be a Lie algebra over \( \mathbb{C} \) and let \( \mathfrak{g}^{(d)} \) be the direct sum of \( d \) copies of \( \mathfrak{g} \). Let \( \theta \) be an automorphism of \( \mathfrak{g} \) and define

\[
\theta_{(d)}(x_1, ..., x_d) = (\theta x_d, x_1, ..., x_{d-1})
\]

then \( \theta_{(d)}^d = (\theta, ..., \theta) \).

**Lemma 12** The map \((x_1, ..., x_d) \mapsto \sum \theta^{d-1} x_i\) defines an isomorphism of the graded Lie algebra \((\theta_{(d)}^{(1)}, (\mathfrak{g}^{(1)})^{(d)})\) onto \((\theta, \tilde{\mathfrak{g}})\).

**Exercise.** Prove this lemma.

### 2.7 Maximal compact groups.

In this subsection we take \( G \) to be semi-simple, connected and such that \( \mathfrak{g} = \text{Lie}(G) \) is \( \mathbb{Z}/m\mathbb{Z} \)-graded. We assume that there exists, \( \tau \), an automorphism of \( G \) such that \( d\tau = \theta \). We assume as, we may, that \( G \) is a \( \mathbb{Z} \)-closed subgroup of \( GL(n, \mathbb{C}) \) that is invariant under adjoint. We set \( K = G \cap U(n) \).

We note that \( \tau(K) \) is a maximal compact subgroup of \( G \) (this is true for any continuous isomorphism). Thus there exists \( g \in G \) such that \( g\tau(K)g^{-1} = K \). Hence, if we define \( \tau'(x) = g\tau(x)g^{-1} \) then \( \tau' \) is an automorphism of \( G \) and
if $\theta'(X) = g\theta(X)g^{-1}$ then $\theta'$ is an element of order $m$ such that $\tau'(e^X) = e^{\theta'X}$.

We will therefore assume that $\tau(K) = K$ and $\theta(\text{Lie}(K)) = \text{Lie}(K)$.

If $\theta^m = 1$ with $m < \infty$, we also fix the inner product

$$\langle X, Y \rangle = \frac{1}{m} \sum_{i=0}^{m-1} \text{tr}(\theta^i X (\theta^i Y)^*)$$

and the symmetric bilinear form

$$\langle X, Y \rangle = \frac{1}{m} \sum_{k=0}^{m-1} \text{tr}(\theta^k X (\theta^k Y)).$$

We note that, since $X \in \text{Lie}(K)$ is skew Hermitian, if $X \neq 0$, $X \in \text{Lie}(K)$ then $\langle X, X \rangle < 0$. This implies that each term in the above sum for $X = Y \neq 0$ in $\text{Lie}(K)$ is negative. Thus $(\ldots, \ldots)$ is negative definite on $\text{Lie}(K)$ and therefore positive definite on $i\text{Lie}(K)$. In particular, $(\ldots, \ldots)$ is non-degenerate on $\text{Lie}(G)$. We also note that since $\theta \text{Ad}(g) \theta^{-1} = \text{Ad}(\tau g)$ for $g \in G$ the form $(\ldots, \ldots)$ is both $G$ and $\theta$–invariant. Similarly, we have an inner product $(\ldots, \ldots)$ that is $K$ and $\theta$–invariant.

We will denote $\text{Lie}(G)$ by $\mathfrak{g}$. Let for $\zeta \in \mathbb{C}$, $\mathfrak{g}_\zeta = \{ X \in \mathfrak{g} \mid \theta X = \zeta X \}$. Then we note that if the order of $\theta$ is $m < \infty$ then

a) $\mathfrak{g}_\zeta = 0$ if $\zeta^m \neq 1$.

b) Set $G^\tau = \{ g \in G \mid \tau(g) = g \}$ then $\text{Lie}(G^\tau) = \mathfrak{g}_1$.

c) If $\theta$ is arbitrary but semisimple then $(\mathfrak{g}_\zeta, \mathfrak{g}_\mu) = 0$ inless $\zeta \mu = 1$ and the corresponding pairing of $\mathfrak{g}_\zeta$ with $\mathfrak{g}_{\zeta^{-1}}$ is perfect. Thus in particular, $\dim \mathfrak{g}_\zeta = \dim \mathfrak{g}_{\zeta^{-1}}$.

**Definition 13** A Vinberg triple $(G, V, \tau)$ consists of a connected semi-simple algebraic group $G$, as above, and automorphism, $\tau$, of $G$ such that $\theta = d\tau$ is semi-simple as an automorphism of $\mathfrak{g} = \text{Lie}(G)$, $V$ is the $\theta$–eigenspace for a nontrivial eigenvalue of $\theta$.

There is also a notion of a $\theta$–space which is a pair $(H, V)$ of a connected algebraic group $H$ and an algebraic $H$–module $V$ such that there is a Vinberg triple $(G, V, \tau)$ such that $H$ is the identity component of $G^\tau$. We note that if $(L, V)$ is a Vinberg pair then if $L$ is non-trivial $(L, V)$ is a $\theta$–space and if $(H, V)$ is a $\theta$–space with $H$ the identity component of $G^\tau$ and $(G, V, \tau)$ is a Vinberg triple then $(\text{Ad}(H)|_V, V)$ is a Vinberg pair.
Lemma 14 If \((H,V)\) is a \(\theta\)-space with \(\theta\) of finite order. Then there exists a Vinberg triple \((G,V,\tau)\) so that if \(g = \text{Lie}(G)\) then

\[ g = \bigoplus_j g_{\zeta^j} \]

with \(d\tau|_{g_{\zeta^j}} = \zeta^j I\) with \(\zeta\) a primitive \(m\)-th root of unity, \(V = g_\zeta\) and \(H\) is the identity component of \(G^\tau\).

**Proof.** By the definition of \(\theta\)-space there \((\tilde{G},\tau,V)\) so that \(H\) is the identity component of \(\tilde{G}^\tau\) and \(V = \text{Lie}(\tilde{G})_\zeta\) with \(\zeta\) is a root of unity. Let \(m\) be the order of \(\zeta\) then \(\zeta\) is a primitive \(m\)-th root of unity. We may assume that \(\tilde{G}\) is contained in \(GL(n,\mathbb{C})\) and is invariant under Hermitian adjoint. Then \(g = \bigoplus_j \text{Lie}(\tilde{G})_{\zeta^j}\) is reductive Lie subalgebra. As above we may assume that \(\tau\) normalizes \(K\) and that there is an inner product on \(\text{Lie}(\tilde{G})\) so that \(d\tau\) and \(\text{Ad}(\tilde{K})\) act unitarily. Let \(K\) be the connected subgroup of \(\tilde{K}\) with \(\text{Lie}(K) = g \cap \text{Lie}(K)\). Since \(\left(\text{Lie}(\tilde{G})_\mu\right)^* = \text{Lie}(\tilde{G})_{-\mu}\) and

\[ \text{Lie}(K) = \{x \in g | x^* = -x\} \]

we have \(\text{span}_\mathbb{C}(\text{Lie}(K)) = g\). We assert that \(K\) is closed. Indeed, let \(U\) be its closure. Then \((\tau|_U)^m = I\). Thus \(\text{span}_\mathbb{C}(U) = \bigoplus_j \text{Lie}(\tilde{G})_{\zeta^j} \cap \text{span}_\mathbb{C}(U)\) since the right hand side is contained in \(g\) and the left hand side contains \(\text{Lie}(K)\) we conclude that \(\text{Lie}(U) = \text{Lie}(K)\) so \(K = U\) (since both groups are connected). Now set \(G\) equal to the Zariski closure of \(U\) in \(\tilde{G}\). Connectedness implies that the identity component of \(G^\tau\) is \(H\). This completes the proof.

**Definition 15** A Vinberg space is a Vinberg triple \((G,\tau,V)\) such that if \(g = \text{Lie}(G)\) then \(g = \bigoplus_j g_{\zeta^j}\) with either \(\zeta\) a primitive \(m\)-th root of unity or \(\zeta\) is of infinite order and \(\text{V} = g_\zeta\).

2.8 A lemma of Richardson

In this section we will describe some ideas of Richardson.

**Lemma 16** Let \(G \subset GL(n,\mathbb{C})\) be algebraic and assume that \(H\) is a Zariski closed connected subgroup of \(G\). Assume that \(V \subset \mathbb{C}^n\) is a \(H\)-invariant subspace such that if \(v \in V\) then \(\text{Lie}(G)v \cap V = \text{Lie}(H)v\). If \(x \in V\) then \(Gx \cap V\) is a disjoint union of a finite number of orbits of \(H\) all of which are of the same dimension and are open in \(Gx \cap V\).
Let \( u \in Gx \cap V \). Then
\[
T_u(Gx \cap V) \subset \text{Lie}(G)u \cap V = \text{Lie}(H)u \cap V.
\]
Also, since
\[
Hu \subset Gu \cap V = Gx \cap V
\]
we see that \( \dim (Gx \cap V) = \dim(Hu) \). This implies that the orbits of \( H \) in \( Gx \cap V \) all have the same dimension and are open in \( Gx \cap V \). Hence they are closed and since they are irreducible they are the irreducible components of the variety \( Gx \cap V \).

**Lemma 17** Let \((G, V, \tau)\) be a Vinberg space and set \( H = G^\tau \). If \( x \in V \) then
\[
[g, x] \cap V = [\text{Lie}(H), x].
\]

**Proof.** If \( y \in g \) then \( y = \sum_{k=0}^{m-1} y_{\zeta^k} \) (as usual). If \( [y, x] \in V \) then \( [y, x] = [y, x]_{\zeta} = [y_1, x] \in \text{Ad}(\text{Lie}(H)).x \). Since it is clear that \([g, x] \cap V \supset [\text{Lie}(H), x]\) the Lemma follows. ■

### 2.9 Basic properties of Vinberg spaces

**1. Infinite order.**

Let \((G, V, \tau)\) be a Vinberg space relative to \( \theta = d\tau \), \( \text{Lie}(G) = g = \bigoplus g_{\zeta^k} \) with \( \zeta \) of infinite order (all but a finite number of the \( g_{\zeta^k} = 0 \)) and \( V = g_\zeta \). We assume that \( K \) is as in the previous section and is \( \tau \)-invariant. As above \( H = G^\tau \). We can think of \( g \) having a \( \mathbb{Z} \)-grading setting \( g_k = g_{\zeta^k} \).

**Lemma 18** If \( x \in V \) then \( \text{ad}(x) \) is nilpotent.

**Proof.** If \( y \in g_{\zeta^k} \) then \( \text{ad}(x)^r y \in g_{\zeta^{k+r}} \). Since only a finite number of \( g_{\zeta^k} \) are non-zero the result follows. ■

Note in the next lemma we won’t assume that \( \zeta \) is of infinite order.

**Lemma 19** If \((G, V, \tau)\) is a Vinberg space then under \( H \) the identity component of \( G^\tau \) the number of nilpotent orbits in \( V \) is finite.

**Proof.** We replace \( G \) with \( \text{Ad}(G) \) and look upon \( G \subset GL(g) \) a Zariski closed subgroup. We note that \( G \) acting on \( g \) has a finite number of nilpotent orbits. The result now follows from Lemmas 17 and 16. ■

We note that this result implies

**Proposition 20** If \((G, V, \tau)\) is a Vinberg space such that every element \( x \in V \) is nilpotent then under \( H \) the identity component of \( G^\tau \) then there are only a finite number of \( H \)-orbits and there exists \( v \in V \) such that \( Hv \) is Zariski open in \( V \).
2.10 Basic properties of Vinberg spaces 2. Finite order

Let \((G, V, \tau)\) be a Vinberg space and assume that \(\tau\) is of order \(m\) and relative to \(\theta = d\tau, Lie(G) = g = \oplus_{k=0}^{m-1} g_{\zeta^k}\) with \(\zeta\) a primitive \(m\)–th root of unity and \(V = g_{\zeta}\). We assume that \(K\) is as in the previous section and is \(\tau\)–invariant.

As above \(H = G^\tau\)

If \(x \in V\) we set \(x = x_s + x_n\) equal to its Jordan decomposition in \(g\). That is, \(x_s\) has the property that it is a diagonalizable element of \(M_n(\mathbb{C})\) (i.e. semisimple), \(x_n\) is nilpotent and \([x_s, x_n] = 0\). This decomposition is unique.

We note that if \(x \in g\) is semisimple then \(x\) is also semisimple similarly if \(x\) is nilpotent \(\theta x\) is nilpotent. Thus \(\theta(x_s) = \theta(x)_{s}\) and \(\theta(x_n) = \theta(x)_n\). Since, \(x \in g_{\zeta}\) we have \(\theta x = \zeta x\) so \(x_s, x_n \in V\).

**Theorem 21** If \(x \in V\) then \(Hx\) is closed if and only if \(x = x_s\).

**Proof.** We first prove that if \(x \in V\) then \(\overline{Hx}\) contains \(x_s\). If \(x_n = 0\) there is nothing to prove. Otherwise, set \(e = x_n\) and let \(f, h \in g_{x_s}\) (that is commute with \(x_s\)) be such that

\[
[e, f] = h, [h, f] = -2f, [h, e] = 2e
\]

Then \(h = \sum_{k=0}^{m-1} h_{\zeta^k}\) with \(\theta h_{\zeta^k} = \zeta^k h_{\zeta^k}\). Thus \(\theta[h, e] = 2\zeta x\) on the one hand and

\[
\theta[h, e] = \sum_{k=0}^{m-1} \zeta^{k+1}[h_{\zeta^k}, e].
\]

This implies that \([h_{\zeta^k}, e] = 0\) unless \(\zeta^k = 1\). Similarly, \([h_{\zeta^k}, x_s] = 0\) for all \(k\). Thus there exists \(u = h_1 \in Lie(H)\) such that \([u, x_s] = 0\) and \([u, x_n] = 2x_n\).

Hence

\[
Ad(e^{-tu})x = x_s + e^{-2t}x_n.
\]

This implies our assertion and therefore proves that if \(x\) isn’t semisimple in \(g\) then \(Hx\) isn’t closed. So if \(Hx\) is closed then \(x\) is semisimple in \(g\).

We now show that if \(x\) is semisimple then \(Hx\) is closed. We note that \(\overline{Hx}\) contains a closed orbit, \(Hu\) with \(u \in V\). We note that \(u \in \overline{Gx} = Gx\) since \(x\) is semisimple in \(g\). Lemmas 17 and 16. above prove that \(\dim Ad(H)x = \dim Ad(H)u\). Since \(Ad(H)u\) is closed, irreducible and contained in \(Ad(H)x\) which is irreducible we see that \(Ad(H)u = Ad(H)x\). ■
3 The structure of a Vinberg pair

In this section we delve into the more delicate aspects of Vinberg's results.

3.1 Cartan Subspaces

Definition 22 Let $(G, V, \tau)$ be a Vinberg space. A Cartan subspace of $V$ is a subspace that consists of semi-simple commuting elements and is maximal subject to this property. The rank of $V$ is the maximum of the dimensions of Cartan subspaces.

Remark 23 The rank of $(G, V, \tau)$ is 0 if and only if every element of $V$ is nilpotent.

For the rest of this section we will assume that $(G, V, \tau)$ is a Vinberg space with $\theta = d\tau$ of order $m$. We will also assume that $G$ is connected.

Fix $a$ a Cartan subspace of $V$. Let $T \subset G$ be the intersection of all Zariski closed subgroups, $U$, of $G$ such that $\text{Lie}(U) \supset a$.

Lemma 24 $T$ is an algebraic torus that is contained in the identity component of the centralizer of $a$ in $G$.

Proof. Consider the $Z$– closure, $T_1$, of $\exp(a)$. Then $T_1$ is abelian and it is contained in every $Z$– closed subgroup that whose Lie algebra contains $a$. Thus since the Lie algebra of a Lie group is the same as the Lie algebra of its identity component we see that $T_1 = T$ and $T$ is connected and abelian. Let $G_1$ be the centeralizer of $a$ in $G$ then $G_1$ is reductive furthermore the center $C$ of $G_1$ has the property that $\text{Lie}(C) \supset a$. Hence $T$ is contained in the identity component of the center of $G_1$. Hence, $T$ consists of semi-simple elements and is connected so it is a torus. ■

We will set $T = T_a$.

Proposition 25 $\dim T_a = \varphi(m) \dim a$ with $\varphi(m)$ the Euler Totient function (the number of $0 < j < m$ with $\gcd(j, m) = 1$).

Proof. Set $T = T_a$. Then $\tau : T \rightarrow T$ is an algebraic automorphism. The character group, $\hat{T}$, of $T$ is a free abelian group of rank equal to $\dim T$. Thus if we identify a character with its differential we find that $\text{Lie}(T)^*$ has a basis $\lambda_1, ..., \lambda_d$ such that $\theta = d\tau$ has an integral matrix. Let $e_1, ..., e_d$ be the
dual basis to $\lambda_1, \ldots, \lambda_d$ then $\theta e_i = \sum a_{ji} e_j$ with $a_{ij} \in \mathbb{Z}$. The characteristic polynomial, $f(x)$, of $\theta|_{\text{Lie}(T)}$ is

$$f(x) = \sum_{k=0}^{d} c_k x^k$$

with $c_j \in \mathbb{Z}$. But since $\theta^m = 1$ we have a factorization over $\mathbb{Q}[[\zeta]]$

$$f(x) = \prod_{k=0}^{m-1} (x - \zeta^k)^{d_k}.$$ 

By definition of $T_a$ and the maximality of $a$, $d_1 = \dim a$. Let for $0 < j < m$ and $\gcd(j, m) = 1$, $\sigma : \mathbb{Q}[[\zeta]] \to \mathbb{Q}[[\zeta]]$ be the element of the Galois group $\mathbb{Q}[[\zeta]]$ over $\mathbb{Q}$ defined by $\sigma(\zeta) = \zeta^j$. Thus

$$f(\sigma(x)) = \sum_{k=0}^{d} c_k \sigma(x)^k = \sigma\left(\sum_{k=0}^{d} c_k x^k\right)$$

also

$$\sigma(f(x)) = \prod_{k=0}^{m-1} (\sigma(x) - \zeta^{kj})^{d_k}$$

and

$$f(\sigma(x)) = \prod_{k=0}^{m-1} (\sigma(x) - \zeta^k)^{d_k}.$$ 

This implies that $d_j = d_1$. Hence, $\dim \text{Lie}(T) \cap g_{\zeta^j} = \dim a$. This yields the lower bound $\dim T \geq \varphi(m) \dim a$.

To prove the upper bound we show that if

$$b = \sum_{\gcd(m, j) = 1} \text{Lie}(T) \cap g_{\zeta^j}$$

then $\exp(b)$ is Zariski closed in $T$. We may assume that $T \subseteq (\mathbb{C}^\times)^n$. We note that if $h$ is the $m$–th cyclotomic polynomial that $b = \ker h(\theta|_{\text{Lie}(T)})$. Let

$$h(x) = \sum_{j=0}^{\varphi(m)-1} r_j x^j.$$ 

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As is standard, $a_j \in \mathbb{Z}$. We consider the regular homomorphism of $T$ to $T$.

$$\beta(z) = \prod_{j=0}^{\varphi(m)-1} \tau(z)^{r_j}. $$

If $z = \exp(u)$ with $u \in \mathfrak{b}$ then

$$\beta(z) = \exp(\sum_{j=0}^{\varphi(m)-1} r_j \theta^j u) = 1. $$

Thus $\ker d\beta$ is equal to $\mathfrak{b}$. Hence the identity component of $\ker \beta$ is an algebraic subtorus of $T$ whose Lie algebra contains $\mathfrak{a}$. This implies that $T = \exp \mathfrak{b}$ and so $\dim T \leq \varphi(m) \dim \mathfrak{a}$. 

### 3.2 Restricted roots

We continue with the notation of the previous section.

Let $\mathfrak{a}$ be a Cartan subspace of $V$ and let $\Sigma(\mathfrak{a})$ be the set elements $\lambda \in \mathfrak{a}^*$, $\lambda \neq 0$ such that there exists $x \in \mathfrak{g}$ such that $x \neq 0$ and $[h, x] = \lambda(h)x$ for all $h \in \mathfrak{a}$. Then $\Sigma(\mathfrak{a})$ is called the restricted root system of the triple. If $\lambda \in \mathfrak{a}^*$ set

$$\mathfrak{g}^\lambda = \{x \in \mathfrak{g} | [h, x] = \lambda(h)x, h \in \mathfrak{a}\}. $$

Then

$$\mathfrak{g} = \mathfrak{g}^{0} \bigoplus_{\lambda \in \Sigma(\mathfrak{a})} \mathfrak{g}^\lambda. $$

Assume that $\Sigma(\mathfrak{a}) = \emptyset$. The space $\mathfrak{a}$ consists of semi-simple elements thus $\mathfrak{a}$ is contained in the center of $\mathfrak{g}$. The maximality in the definition of a Cartan subspace implies that $\mathfrak{a}$ is the unique Cartan subspace. Also, since every semi-simple element in $V$ is contained in some Cartan subspace we see that every semi-simple element in $V$ is in $\mathfrak{a}$. Lemma 24 implies that $T = T_\mathfrak{a}$ is central. Setting $G' = G/T$ and $\mathfrak{g}' = \text{Lie}(G')$ then $(G', V/\mathfrak{a}, \tau_{(G')})$ is a Vinberg triple of rank 0. We eliminate this complication by adhering to the following definition:

**Definition 26** A Vinberg triple will be called semi-simple if $G$ is semi-simple (that is, $\text{Lie}(G)$ has 0 center). We will abbreviate this to SS Vinberg triple.
Lemma 27 Assume that \((G, V, \tau)\) is a semi-simple Vinberg triple. If \(a\) is a non-zero Cartan subspace of \(V\) then the set \(\Sigma(a)\) spans the dual space of \(a\).

Proof. If \(h \in a\) satisfies \(\Sigma(a)(h) = \{0\}\) then since \(h\) is semi-simple \([h, g] = 0\). Thus \(h = 0\). □

For the rest of this section we will assume that the rank of \((G, V, \tau)\) is semi-simple and not 0. Fix for the moment a Cartan subspace, \(a\). Then an element, \(h\), of \(a\) is called regular if \(\lambda(h) \neq 0\) for all \(\lambda \in \Sigma(a)\). We set \(a'\) equal to the set of regular elements of \(a\).

If \(S \subset V\) is a set of commuting semi-simple elements then we set \(G^S = \{g \in G|gs = s, s \in S\}\). Then \(G^S\) is a reductive algebraic group and

\[
(G^S, \text{Lie}(G^S) \cap V, \tau_{|G^S})
\]

is a Vinberg triple. We note the \(\text{Lie}(G^S) = \{x \in \mathfrak{g}|[x, s] = 0, s \in S\} = C_{\mathfrak{g}}(S)\).

Lemma 28 If \(x \in C_{\mathfrak{g}}(a)_\zeta\) then \(x_s \in a\).

Proof. We have seen that \(x_s \in V\). Furthermore, if \([x, h] = 0\) then \([x_s, h] = 0\). The lemma follows. □

Lemma 29 Let \(h \in a'\) then \(C_{\mathfrak{g}}(h)_\zeta = C_{\mathfrak{g}}(a)_\zeta\).

Proof. This follows since the restricted root decomposition implies \(C_{\mathfrak{g}}(a) = \mathfrak{g}^0 = C_{\mathfrak{g}}(h)\). □

Proposition 30 Let \(H\) be the identity component of \(G^\tau\). Set

\[
r(a) = \{x \in C_{\mathfrak{g}}(a)_\zeta|x_s \in a'\}.
\]

Then \(\text{Ad}(H)r(a)\) has non-empty Zariski interior in \(V\).

Proof. We first prove that the map

\[
\Phi : H \times r(a) \to V
\]

\[
(h, x) \mapsto \text{Ad}(h)x
\]

has surjective differential for all \((h, x) \in H \times r(a)\). Let \((..., ...)\) be the form defined in section 2.2 then both \(G\) and \(\theta\) leave it invariant. This implies
that it induces a perfect pairing between $V = g_\xi$ and $g_{\xi^{-1}}$. We note that if $x, v \in r(a)$ and $h \in H$ then if $X \in \text{Lie}(H)$ then

$$d\Phi_{h,x}(X_h, v) = \text{Ad}(h)([X, x] + v).$$

If this differential is not surjective then there would exist $u \in g_{\xi^{-1}}$ such that

$$(u, [X, x] + v) = 0$$

for all $v \in r(a)$ and $X \in \text{Lie}(H) = g_1$. This implies that

$$0 = (u, [X, x]) = ([x, u], X)$$

so $[u, x]$ is orthogonal to $g_1$. So since $[u, x] \in g_1$ this implies that $[u, x] = 0$. As before, $\ker ad x \subset \ker ad x_s$ which $u \in C_g(x_s) = C_g(a)$. Hence $u \in C_g(a)_{\xi^{-1}}$ and is orthogonal to $C_g(a)_{\xi}$. Thus $u = 0$. This implies that $\Phi$ has open image in the Euclidian topology. Thus the image of $\Phi$ is Zariski dense and so has non-zero Zariski interior. 

**Corollary 31** If $a$ and $b$ are Cartan subspaces of $V$ then there exists $h \in H$ such that $ha = b$.

**Proof.** The sets $Ad(H)r(a)$ and $Ad(H)r(b)$ both have non-empty Zariski interior. This implies that they must have non-trivial intersection. Thus there is $h \in H, x \in r(a)$ such that $Ad(h)x \in r(b)$. Thus $Ad(h)x_s \in b$. So $Ad(h)a \subset b$. The maximality implies equality. 

**Corollary 32** Let $a$ be a Cartan subspace of $V$. If $x \in V$ and $Ad(H)x$ is closed in $V$ then there exists $h \in H$ such that $Ad(h)x \in a$.

**Proof.** If $b$ is maximal in $V$ subject to the conditions that pairs of elements in $b$ commute, every element in $b$ is semisimple and $x \in b$. Then $b$ is a Cartan subspace of $V$ so the previous corollary implies that there is an $h \in H$ such that $Ad(h)b = a$. 

**Corollary 33** Let $a$ be a Cartan subspace of $V$. Then $Ad(H)r(a)$ is Zariski open in $V$.
Proof. Let for $X \in V$

$$\det(tI - adX) = \sum_{j=0}^{p} t^j D_{p-j}(X)$$

with $p = \dim g$. Since each $D_{p-j}$ is $H$ invariant $D_{p-j}(X) = D_{p-j}(X_s)$. There exists $h \in H$ and $x \in a$ such that $X_s = Ad(h)x$. For $x \in a$ we have

$$\det(tI - adX) = t^{\dim g_0} \prod_{\lambda \in \Sigma(a)} (t - \lambda(x))^{\dim g^\lambda}.$$ 

Set $u = \dim g^0$. Thus if $x \in a$ then $D_{p-j} = 0$ for $j < u$ and

$$D_{p-u}(x) = (-1)^{p-u} \prod_{\lambda \in \Sigma(a)} \lambda(x)^{\dim g^\lambda}.$$ 

This implies that $Ad(H)r(a) = \{x \in V | D_{p-u}(x) \neq 0\}$. ■

3.3 The Weyl group

If $G$ is a Lie group, $H$ is a closed subgroup and $a$ is a subspace of $Lie(G)$ then the normalizer of $a$ in $H$ is the subgroup

$$N_H(a) = \{g \in H | Ad(g)a \subset a\}$$

and the centralizer of $a$ is the subgroup

$$C_H(a) = \{g \in H | Ad(g)x = x, x \in a\}.$$ 

The Weyl group of $a$ is the group

$$W_H(a) = N_H(a)/C_H(a)$$

which we will think of as a group of linear maps of $a$ to $a$.

Lemma 34 If $adx$ is diagonalizable for every element in $a$ and $[x, y] = 0, x, y \in a$ then $Lie(N_H(a)) = Lie(C_H(a))$. 

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Proof. We note that

\[ \operatorname{Lie}(N_H(\mathfrak{a})) = \{ x \in \operatorname{Lie}(H) | \text{ad}(x)a \subset \mathfrak{a} \} \]

and

\[ \operatorname{Lie}(C_H(\mathfrak{a})) = \{ x \in \operatorname{Lie}(H) | \text{ad}(x)a = 0 \}. \]

If \( x \in \operatorname{Lie}(N_H(\mathfrak{a})) \) and \( y \in \mathfrak{a} \) then \([x, y] \in \mathfrak{a} \) so \((\text{ad}y)^2 x = 0\). This implies that \([y, x] = 0\) since \(\text{ad}y\) is diagonalizable. Hence \( y \in \operatorname{Lie}(C_H(\mathfrak{a})) \).

**Corollary 35** If \( \mathfrak{a} \) satisfies the conditions of the previous lemma and if \( G \) is a linear algebraic group and \( H \) is a \( \mathbb{Z} \)-closed subgroup then \( W_H(\mathfrak{a}) \) is finite.

**Proof.** The previous lemma implies that \( C_H(\mathfrak{a}) \) contains the identity component of \( N_H(\mathfrak{a}) \) since an algebraic group has a finite number of connected components the result follows.

We now return to the situation of the previous section.

**Corollary 36** Let \( \mathfrak{a} \) be a Cartan subspace of \( V \) then \( W_H(\mathfrak{a}) \) is a finite group which we call the Weyl group of the Vinberg space.

We note that since any two Cartan spaces of \( V \) are conjugate by \( H \) the Weyl group is, up to isomorphism, independent of the choice of \( \mathfrak{a} \).

**Proposition 37** If \( x, y \in \mathfrak{a} \), a Cartan subspace of \( V \) then \( \text{Ad}(H)x = \text{Ad}(H)y \) if and only if \( W_H(\mathfrak{a})x = y \).

**Proof.** We assume \( x, y \in \mathfrak{a} \) and that there exists \( h \in H \) with \( hx = y \). We consider the Vinberg triple \((C_G(y), \operatorname{Lie}(C_G(y)) \cap V, \tau_{|C_G(y)})\). Then \( \mathfrak{a} \) and \( \text{Ad}(h)\mathfrak{a} \) are Cartan subspaces of \( \operatorname{Lie}(C_G(y)) \cap V \). Hence there exists \( u \) in the identity component of \( C_G(y)^\tau \) such that \( \text{Ad}(u)\text{Ad}(h)\mathfrak{a} = \mathfrak{a} \). We have \( \text{Ad}(uh)x = \text{Ad}(u)y = y \). Thus \( uh \in N_H(\mathfrak{a}) \) and \( \text{Ad}(uh)x = y \).

We now come to Vinberg’s generalization of the Chevalley restriction theorem. Our proof is a bit simpler than the original.

**Theorem 38** Assume that \( G \) is semi-simple. The restriction map \( \text{res}_{V/\mathfrak{a}} : \mathcal{O}(V)^H \to \mathcal{O}(\mathfrak{a})^{W_H} \) is an isomorphism of algebras.
Proof. Set $W = W_H(a)$. We first observe

1. If $x, y \in a$ and $Wx \capWy = \emptyset$ then there exists $f \in \mathcal{O}(V)^H$ with $f(x) = 0$ and $f(y) = 1$.

Indeed, if $x, y \in a$ then $Ad(H)x$ and $Ad(H)y$ are $\mathbb{Z}$-closed in $a$. The previous proposition implies that $Ad(H)x \cap Ad(H)y = \emptyset$. Theorem 36 in [GIT] asserts the existence of $f$.

Set $B = \text{res}_{V/a}(\mathcal{O}(V)^H)$ and let $q(B)$ be the quotient field of $B$. Let $F$ be the field of rational functions on $a$. Then $F$ is the quotient field of $\mathcal{O}(a)$. We assert that

2. $F$ is a normal extension of $q(B)$.

Consider,

$$\det(tI - \text{ad}X) = \sum_{j=0}^{p} t^j D_{p-j}(X)$$

for $X \in V$ (see the previous sub-section) then $D_{p-j} \in \mathcal{O}(V)^H$ so $h(t) = \sum_{j=0}^{p} t^j \text{res}_{V/a} D_{p-j}(X)$ is in $q(B)[t]$. The roots of this polynomial are 0 and the elements of $\Sigma(a)$. Since the span of $\Sigma(a)$ is $a^*$ we see that $F$ is the splitting field of $h(t)$ and thus a normal extension of $q(B)$. This implies that

$$q(B) = \{f \in F | \sigma f = f, \sigma \in \text{Gal}(F/q(B))\}.$$

Also,

3. If $\sigma \in \text{Gal}(F/q(B))$ then $\sigma(a^*) = a^*$ hence $\sigma \mathcal{O}(a) = \mathcal{O}(a)$.

Let $U = \{\sigma \in \text{Aut}(\mathcal{O}(a)) | \sigma|_B = I\}$. Then $U = \text{Gal}(F/q(B))|_{\mathcal{O}(a)}$. We now come to the crux of the argument. Let $\sigma \in U$ then we set for $x \in a$, $f \in \mathcal{O}(a)$

$$\delta_{\sigma,x}(f) = \sigma f(x).$$

This defines a homomorphism of $\mathcal{O}(a)$ to $\mathbb{C}$. The Nullstellensatz implies that there exists $x_1 \in a$ such that $\delta_{\sigma,x}(f) = f(x_1)$ for all $f \in \mathcal{O}(a)$. By the definition of $U$ we see that $f(x_1) = f(x)$ for all $f \in B$. 1. implies that there exists $s \in W$ so that $x_1 = sx$. Hence if $f \in \mathcal{O}(a)^W$ then $\sigma f = f$. This implies that $\mathcal{O}(a)^W \subset B$. Since the converse is obvious the result follows. ■

3.4 The GIT quotient

Let $(H, V)$ be a Vinberg pair corresponding to a graded Lie algebra, $\mathfrak{g} = \bigoplus_{k=0}^{m-1} \mathfrak{g}_k$ with $\zeta$ a primitive $m$-th root of unity. Then $\mathcal{O}(V)^H$ is a Noetherian algebra and the variety $\text{spec}_{\text{max}} \mathcal{O}(V)^H$ is an affine variety denoted $V//H$
(the categoric or GIT quotient). Since $\mathcal{O}(V)^H$ is a subalgebra of $\mathcal{O}(V)$ we have a morphism $\Phi : V \to V//H$. Let $a$ be a Cartan subspace of $V$ and let $l = \dim a = \text{rank}(g)$.

**Theorem 39** Each fiber of $\Phi$ is connected, of codimension $l$ and has a finite number of $H$–orbits. In particular, each fiber has a unique closed orbit and a unique open orbit.

**Proof.** We note that the fiber $\Phi^{-1}(\Phi(0))$ is the set of nilpotent elements in $V$ which decomposes into a finite number of $H$–orbits (Theorem 19). We note that $\Phi(x) = \Phi(y)$ if and only if $Hx_s = Hy_s$. Indeed, consider the set $F_x = \Phi^{-1}(\Phi(x))$. We note that if $y \in F_x$ then $f(x) = f(y)$ for all $f \in \mathcal{O}(V)^H$. This implies that $f(x_s) = f(y_s)$ for all $f \in \mathcal{O}(V)^H$. This implies that $Hx_s = Hy_s$. We also have $x = x_s + y_s$ with $[x_s, y_s] = 0$. Thus the set of elements in $F_x$ is precisely the collection of elements in $V$ of the form $h(x_s + z)$ with $z$ nilpotent and in $C_g(x_s) \cap V$. Indeed, if $y \in F_x$ we have seen that $y_s = hx_s$ for some $h \in H$. Thus

$$h^{-1}y = x_s + h^{-1}y_s = x_s + z$$

implying the assertion. But $(Hx_s, C_g(x_s) \cap V)$ is a Vindberg pair. This implies that the set of such $z$ breaks up into a finite number of $Hx_s$ orbits.

Let $V''$ be the set of points $v \in V$ such that $\dim \text{Ad}(H)v$ is of maximal dimension. Then $V''$ is Zariski open and dense in $V$. This implies that the maximal dimension is the generic dimension. Since each fibre of $\Phi$ contains an open orbit and the generic dimension of the fiber of $\Phi$ is the minimal dimension (see, for example, Shafarevich, Basic Algebraic Geometry, p.60, Theorem 1.3.7) we must have all fibers have the same dimension. We note that the dimension of $V//H$ is the same as that of $a/W$ which is $l$. Thus the minimal dimension of a fiber is $\dim V - l$. 

3.5 Complex reflections

We assume that $(G, V, \tau)$ is a semi-simple Vinberg Space with $\theta = d\tau$ and $\theta^n = 1$. Let $a$ be a Cartan subspace of $V$ and let $\Sigma(a)$ be the root system of $a$. Let $\lambda \in \Sigma(a)$ and set $g^{[\lambda]}$ equal to the

$$g^0 \oplus \bigoplus_{\mu \in \Sigma(a) \cap \mathbb{C}A} g^\mu.$$

We now collect some properties of $g^{[\lambda]}$. 21
Lemma 40 The subalgebra \( g^{[\lambda]} \) is invariant under \( \theta \). In fact, \( \theta g^\mu = g^\zeta^{-1} \mu \).

**Proof.** The last assertion implies the first. Let \( y \in g^\mu \) and let \( x \in a \). Then
\[
[x, \theta y] = \theta[\theta^{-1} x, y] = \zeta^{-1} \mu(x) \theta y.
\]

\( \blacksquare \)

Lemma 41 \( g^{[\lambda]} \) is reductive.

**Proof.** Let \( B \) denote the Killing form of \( g \). Then we have \( B(g^\mu, g^\nu) = 0 \) unless, \( \mu + \nu = 0 \). This implies that \( B \) is nondegenerate on \( g^{[\lambda]} \). \( \blacksquare \)

Lemma 42 Let \( y \in g^\mu \) and write \( y = \sum_{j=0}^{m-1} y_{\zeta^j} \) relative to the grade on \( g \) corresponding to \( \theta \) then if \( x \in a \) and \( \mu(x) \neq 0 \) then
1. \( \left( \frac{ad(x)}{\mu(x)} \right)^m y_1 = y_1 \).
2. \( y_{\zeta^j} = \left( \frac{ad(x)}{\mu(x)} \right)^j y_1, j = 0, ..., m - 1 \).

**Proof.** \( \lambda(x)y = \sum_j ad(x) y_{\zeta^j} \). Since \( x \in g_\xi \), \( ad(x) y_{\zeta^j} = \lambda(x) y_{\zeta^j+1} \). This implies the result. \( \blacksquare \)

We consider, \([g^{[\lambda]}, g^{[\lambda]}] \) which is a semi-simple Lie algebra that is \( \mathbb{Z}/m\mathbb{Z} \) graded. We also note that \( a \cap [g^{[\lambda]}, g^{[\lambda]}] \neq 0 \). Also, the restricted root system of \( g^{[\lambda]} \) on \( a \) consists of multiples of \( \lambda \). Thus we see that Lemma 27 imples that
\[
\dim ([g^{[\lambda]}, g^{[\lambda]}] \cap a) = 1.
\]
We have proved

Lemma 43 There exists a unique element of \( H_\lambda \in [g^{[\lambda]}, g^{[\lambda]}] \cap a \) such that \( \lambda(H_\lambda) = 1 \).

Proposition 44 Let \((G, V, \tau)\) be a rank one semi-simple Vinberg space then its Weyl group is a non-trivial cyclic group.

**Proof.** Let \((L, V)\) be the corresponding Vinberg pair. Lemma 11 implies that \( V \) is isomorphic with a direct sum of simple Vinberg pairs. Since the rank is one only one of the pairs has positive rank, which is one. Thus we can assume that \( g = \text{Lie}(G) \) is simple and that \( \theta = d\tau \) has order \( m \). Let \( a \) be a Cartan subspace of \( V \). Then since the rank is 1, \( \Sigma(a) \subset \mathbb{C}\lambda \) with \( \lambda \) a restricted root. This implies that the Weyl group is isomorphic with a subgroup of \( \mathbb{C}^\times \). Thus
it is a cyclic group (see the appendix to this subsection). Assume that the group is trivial. We will derive a contradiction. Since $O(a)^W = O(a)$ we see that $O(V)^H = \mathbb{C}[u]$ with $u \in V^*$. Let $H_\lambda \in a$ be defined by $\lambda(H_\lambda) = 1$. Since $H_\lambda$ is semi-simple and $a = \mathbb{C}H_\lambda$ we may assume that $u(H_\lambda) = 1$. We also note that ker $u$ is $H$-invariant thus complete reducibility implies that

$$V = \mathbb{C}v \oplus \ker u$$

with $Ad(H)v \subset \mathbb{C}v$. We also note that since $u$ generates the $H$–invariants ker $u$ is precisely the set of nilpotent elements in $V$. Let $Ad(h)v = \chi(h)v$ be the action of $H$ on $\mathbb{C}v$. Let $v = v_s + v_n$ be its Jordan decomposition. If $h \in H$,

$$Ad(h)v_s = (Ad(h)v)_s = \chi(h)v_s.$$ 

Since $u(v) \neq 0, v_s \neq 0$. So we may assume $v = v_s$. There exists $h \in H$ such that $Ad(h)v = cH_\lambda$ thus

$$\chi(h)v = cH_\lambda.$$ 

We therefore see that we may assume that $v = H_\lambda$. Since the Weyl group is trivial we see that $\chi$ is the trivial character. We are finally ready for the contradiction. Let $y \in g^\lambda$ and let $y = \sum_{k=0}^{m-1} y_{\zeta^k} (\zeta$ a primitive $m$–th root of unity) relative to the grade. Lemma 42 implies that

$$y_{\zeta^k} = ad(H_\lambda)^k y_1$$

and $ad(H_\lambda)^m y_1 = y_1$. But $Ad(H)$ fixes $H_\lambda$. Hence $y = 0$. This is the desired contradiction.

**Definition 45** If $X$ is a complex vector space then a complex reflection is a linear transformation $s : X \to X$ such that $\ker(s - I)$ has codimension 1 and it has a non-trivial root of unity as an eigenvalue.

**Proposition 46** Let $(G,V,\tau)$ be a semi-simple Vinberg space and let $a$ be a Cartan subspace of $V$ with $\Sigma(a)$ the corresponding root system and let $H$ be the identity component of $G^\tau$. Let $\lambda \in \Sigma(a)$ then there exists a complex reflection $s_\lambda \in W_H(a) = W$ such that $\ker(s_\lambda - I) = \ker \lambda$ and the order of $s_\lambda$ is the order of the Weyl group of $g^{[\lambda]}$.

**Proof.** We note that if $y \in g^{[\lambda]}$ and if $x \in a$ and $\lambda(x) = 0$ then $[x,y] = 0$. Let $G^{[\lambda]}$ be the connected group corresponding to $g^{[\lambda]}$ and let $H^{[\lambda]}$ be the
identity component of $H \cap G^{[\lambda]}$. Then we know that $W_{H^{[\lambda]}}([g^{[\lambda]}], g^{[\lambda]}] \cap \mathfrak{a}$ is a cyclic group. Let $s$ be a generator and let $l$ be its order. Let $h \in H^{[\lambda]} \subset H$ be a representative. Then $Ad(h)H_\lambda = \mu H_\lambda$ with $\mu$ a primitive $l$–th root of unity. Also, $Ad(h)|_{\ker \lambda} = I$. Thus $Ad(h)|_{\mathfrak{a}}$ is a complex reflection of order $l$.

**Lemma 47** We continue the notation in the previous proposition. If $\lambda \in \Sigma(\mathfrak{a})$ and $x \in \mathfrak{a}$ then

$$s_\lambda(x) = x + (\mu - 1)\lambda(x)H_\lambda$$

with $\mu$ a primitive $l$–th root of unity.

**Proof.** $\mathfrak{a} = \mathbb{C}H_\lambda \oplus \ker \lambda$. If $x \in \mathfrak{a}$ then $x = \lambda(x)H_\lambda + (x - \lambda(x)H_\lambda)$ is its expression relative to the direct sum decomposition. Thus

$$s_\lambda(x) = \mu \lambda(x)H_\lambda + x - \lambda(x)H_\lambda.$$

We note that the reflection $s_\lambda$ depends on the choice of primitive $l$–root of unity but otherwise is the same for any multiple of $\lambda$ that is a root.

**Proposition 48** We maintain the notation of the Proposition above. Let $\tilde{W}$ be the subgroup of $W$ generated by the reflections $s_\lambda$ for $\lambda \in \Sigma(\mathfrak{a})$. Then $\tilde{W}$ is a normal subgroup of $W$. If $\mathfrak{g}$ is simple then $\tilde{W}$ acts irreducibly on $\mathfrak{a}$.

**Proof.** If $s \in W$ and $\lambda \in \Sigma(\mathfrak{a})$ then $s\lambda = \lambda \circ s^{-1} \in \Sigma(\mathfrak{a})$. This an the previous Lemma implies that $\tilde{W}$ is normal. Suppose that $\mathfrak{g}$ is simple of $\tilde{W}$ is and $\mathfrak{a} = A \oplus B$ with $\tilde{W}A = A$ and $\tilde{W}B = B$. If $a \in A$ then $s_\lambda(a) = (l - 1)\lambda(a)H_\lambda + a$. Thus if $\lambda(a) \neq 0$ then $H_\lambda \in A$. This implies that if $\lambda|_A \neq 0$ then $\lambda|_B = 0$ and vice-versa. Set $\Sigma_A = \{ \lambda \in \Sigma(\mathfrak{a})|\lambda|_A \neq 0 \}$ and $\Sigma_B = \{ \lambda \in \Sigma(\mathfrak{a})|\lambda|_B \neq 0 \}$. Then $\Sigma(\mathfrak{a}) = \Sigma_A \cup \Sigma_B$ a disjoint union (since $\Sigma(\mathfrak{a})$ spans the dual space of $\mathfrak{a}$). We also note that the above observation implies that if $\lambda \in \Sigma_A$ and $\mu \in \Sigma_B$ then $c\lambda + d\mu \in \Sigma(\mathfrak{a})$ only if $cd = 0$. Let

$$Z_A = \sum_{\lambda \in \Sigma_A} \mathfrak{g}^\lambda, Z_B = \sum_{\lambda \in \Sigma_B} \mathfrak{g}^\lambda$$

and let $\mathfrak{g}_A$ (resp. $\mathfrak{g}_B$) be the Lie subalgebra of $\mathfrak{g}$ generated by $Z_A$ (resp. $Z_B$).

Noting that $[Z_A, Z_B] = 0$ we see that $\mathfrak{g}_A$ and $\mathfrak{g}_B$ are ideals in $\mathfrak{g}$. Thus one of them must be 0. Hence $A$ or $B$ is 0.

We finally come to the crux of the matter.
3.5.1 Appendix

The purpose of this appendix is to prove two technical Lemmas needed for the subsection above and to recall a standard result with a reference. The first uses an argument that was posted by Andrea Petracci on the internet.

**Lemma 49** If $G$ is a finite subgroup of the multiplicative group of a field then $G$ is cyclic.

**Proof.** We note that this result follows from the following observation:

Let $G$ be a finite group with $n$ elements. If for every $d|n$ we have $|\{x \in G| x^d = 1\}| \leq d$, then $G$ is cyclic.

Indeed, if $F$ is a field then the number of solutions of a polynomial of degree $d$ is at most $d$.

We now prove the observation. Fix $d|n$ and consider the set $G_d$ consisting of the elements of $G$ with order $d$. Suppose that $G_d \neq \emptyset$ and let $y \in G_d$. Then

$$\langle y \rangle = \{y^j | j \in \mathbb{Z} \} \subset \{x \in G| x^d = 1\}.$$  

But the subgroup $\langle y \rangle$ has cardinality $d$ so $\langle y \rangle = \{x \in G| x^d = 1\}$. Therefore $G_d$ is the set of generators of the cyclic group $\langle y \rangle$ of order $d$. We have shown that $|G_d| = \varphi(d)$ with $\varphi(d)$ Euler’s totient function (the number of positive integers less than $d$ relatively prime to $d$).

We have proved that $G_d$ is empty or has cardinality $\varphi(d)$, for every $d|n$. So we have

$$n = |G| = \sum_{d|n} G_d \leq \sum_{d|n} \varphi(d) = n.$$  

Therefore $|G_d| = \varphi(d)$ for every $d$ dividing $n$. In particular $G_d \neq \emptyset$. This proves that $G$ is cyclic. ■

The next result is due to Vinberg.

**Lemma 50** Let $V$ be a finite dimensional vector space over $\mathbb{C}$ and assume that $G$ is a finite subgroup of $GL(V)$ such that $(V - \{0\})/G$ is simply connected in the quotient topology. $G$ is generated by its elements that have non-zero fixed points.

**Proof.** Let $H$ be the subgroup of $G$ generated by those elements with non-zero fixed points. Then $H$ is a normal subgroup of $G$. Hence

$$(V - \{0\})/G = ((V - \{0\})/H)/(G/H).$$
So \((V - \{0\})/H\) is a covering space of \((V - \{0\})/G\) with the group of deck transformations \(G/H\). But \((V - \{0\})/G\) is simply connected so we must have \(G/ = H\). □

The standard result is

**Proposition 51** Let \(G\) be reductive and connected. Assume that \(x \in \text{Lie}(G)\) is semi-simple. Then \(C_G(x)\) is connected.

**Proof.** Let \(T\) be the Zariski closure of \(\{e^{tx} | t \in \mathbb{C}\}\) in \(G\). Then \(T\) is an algebraic torus. The result now follows from Corollary 11.12 in [B] page 152. □