1 The groups and Lie algebras

We consider $G \subset GL(n, \mathbb{C})$ ($n \times n$ invertible matrices) a subgroup given as the locus of zeros of polymomial in $\mathbb{C}[x_{ij}]$ with x_{ij} the matrix entries of an $n \times n$–matrix that is an element of $M_n(\mathbb{C})$. G in the jargon of algebraic group theory is an affine algebraic group, i.e. a Zariski closed subgroup of $GL(n, \mathbb{C})$. Set X^T equal to the transpose of X and $X^* = \bar{X}^T$ as usual.

Let $Lie(G)$ denote its Lie algebra which we can take to be all $X \in M_n(\mathbb{C})$ such that $e^{tX} \in G$ for all $t \in \mathbb{C}$. Here if $\|X\| = tr(XX^*)^{\frac{1}{2}}$ then $\|...\|$ defines a norm on the vector space $M_n(\mathbb{C})$ and

$$\|XY\| \leq \|X\| \, \|Y\| \, , X, Y \in M_n(\mathbb{C}).$$

Thus the series

$$e^X = \sum_{m=0}^{\infty} \frac{X^m}{m!}$$

converges uniformly and absolutely on closed balls $\|X\| \leq r$. This implies that $X \mapsto e^X$ defines a complex analytic function from $M_n(\mathbb{C})$ to the open subset $GL(n, \mathbb{C})$. It is standard that if $X, Y \in Lie(G)$ then

$$[X, Y] = XY - YX \in Lie(G).$$

As usual, we set $U(n) = \{g \in M_n(\mathbb{C}) | g^* g = I\}$. Then $U(n)$ is a compact subgroup of $GL(n, \mathbb{C})$(it is a closed subset of the unit sphere in $M_n(\mathbb{C})$) and so is a Lie subgroup. Its Lie algebra is precisely the set of skew-hermitian $n \times n$ matrices.

If we assume that if $g \in G$ then $g^* \in G$, that is, G is a symmetric subgroup. We set $K = G \cap U(n)$. Then K is also a compact Lie group and $Lie(K)$ is the set of all elements in $Lie(G)$ that are skew-Hermitian.

Theorem 1 *(cf. [GIT,Theorem 37])Let H be a Zariski closed, reductive subgroup of, G, a symmetric subgroup of $GL(n, \mathbb{C})$ then there exists $g \in G$ such that gHg^{-1} is symmetric.*

Thus we can take as a definition of reductive that there exists an imbedding in $GL(n, \mathbb{C})$ as a symmetric subgroup.

We assume that G is symmetric. A basic structural result is

$$Lie(G) = Lie(K) \oplus iLie(K)$$

as a real vector space and the map

$$K \times iLie(K) \to G$$
$$k, X \mapsto ke^X$$

is a diffeomorphism onto G. Clearly this implies that K is a maximal compact subgroup of G. One can prove that all maximal compact subgroups of G are conjugate to K. In particular, any compact subgroup of G can be conjugated into K.

As usual we denote by $Ad(g)X$ the action of the inner automorphism of $Lie(G)$ corresponding to $g \in G$ on X. That is, $Ad(g)X = gXg^{-1}$. We also write $Inn(g)x = gxg^{-1}$ for $g, x \in G$.

On $Lie(G)$ we put the bilinear form $(X, Y) = \mathrm{tr}XY$. We note that $(..., ...)$ is G-invariant, that is,

$$(gXg^{-1}, gXg^{-1}) = \mathrm{tr}(gXYg^{-1}) = \mathrm{tr}(XY) = (X, Y).$$

Then since G is invariant under $g \longmapsto g^*$ (hence so is $Lie(G)$) the form is nondegenerate.

Lemma 2 *If $H \subset G$ is Zariski closed and reductive then $(..., ...)_{|Lie(H)}$ is non-degerate.*

Proof. We have observed above that there exists $g \in G$ so that gHg^{-1} is invariant under $h \longmapsto h^*$. Hence the form is nondegenerate on $gLie(H)g^{-1}$. The G–invariance of the form implies the result. ■

We note that the converse is also true. Here are examples of reductive groups

1. $GL(n, \mathbb{C})$.
2. $SL(n, \mathbb{C}) = \{g \in GL(n, \mathbb{C}) | \det(g) = 1\}$.
3. $O(n, \mathbb{C}) = \{g \in GL(n, \mathbb{C}) | gIg^T = I\}$.
4. $SO(n, \mathbb{C}) = O(n, \mathbb{C}) \cap SL(n, \mathbb{C})$.
5. Let

$$J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$$

with I the $n \times n$ identity matrix. Set $Sp(n, \mathbb{C}) = \{g \in GL(n, \mathbb{C}) | gJg^T = J\}$.

Exercise. These five groups and arbitrary finite products of them are symmetric subgroups.

These groups and their coverings and quotients give the full set of classical groups.

2 Gradings of Lie algebras

Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{C}$ and let S be a commutative semi-group (written additively) then an S–grading of $\mathfrak{g}$ is a direct sum decomposition

$$\mathfrak{g} = \bigoplus_{s \in S} \mathfrak{g}_s$$

with $\mathfrak{g}_s$ possibly 0 and

$$[\mathfrak{g}_s, \mathfrak{g}_t] \subset \mathfrak{g}_{s+t}$$

for $s, t \in S$.

2.1 $\mathbb{Z}$–gradings

The main examples that are of interest to us are when $S = \mathbb{Z}$ or $S = \mathbb{Z}/n\mathbb{Z}$ with n a stricly positive integer. We will only be looking at finite dimensional Lie algebras.

We have the following characterization of a $\mathbb{Z}$–grading.

Lemma 3 *A $\mathbb{Z}$–grading of $\mathfrak{g}$ is equivalent to having an algebraic homomorphism $\phi : \mathbb{C}^\times \to Aut(\mathfrak{g})$ via $\phi(z)x = z^n x$ for $x \in \mathfrak{g}_n$.*

Proof. We leave it to the reader to check that the formula in the statement defines an algebraic homomorphism of $\mathbb{C}^\times$ to $Aut(\mathfrak{g})$. If ϕ is an algebraic homomorphism of $\mathbb{C}^\times$ to $Aut(\mathfrak{g})$ then $\mathfrak{g}$ is the direct sum of spaces $\mathfrak{g}_\chi$ with $\phi(z)x = \chi(z)x$ for $x \in \mathfrak{g}_\chi$ with χ an algebraic homomorphism of $\mathbb{C}^\times$ to itself (i.e. an algebraic character). The algebraic characters of $\mathbb{C}^\times$ are exactly the ones given by $\chi_n(z) = z^n$. If we set $\mathfrak{g}_n = \mathfrak{g}_{\chi_n}$ we have our grade. ■

Exercises. 1.When is a $\mathbb{Z}/n\mathbb{Z}$ grading a $\mathbb{Z}$–grading?

2. Let $k > 0$ be an integer and let $\mathfrak{g} = \bigoplus_{s \in \mathbb{Z}} \mathfrak{g}_s$ be a $\mathbb{Z}$–grading of $\mathfrak{g}$. Let for $n \in \mathbb{Z}$, $[n]$ be its class mod k. Then defining $\mathfrak{g}_{[n]} = \bigoplus_{r \equiv n \bmod k} \mathfrak{g}_r$ we have a $\mathbb{Z}/n\mathbb{Z}$ grading.

2.2 $\mathbb{Z}/n\mathbb{Z}$ gradings

We now consider the case when we have a $\mathbb{Z}/n\mathbb{Z}$-grading. Then if ζ is an n–th root of unity we can define $\theta : \mathfrak{g} \to \mathfrak{g}$ by $\theta_{|\mathfrak{g}_{[m]}} = \zeta^m I$. Then θ defines an automorphism of $\mathfrak{g}$ of order n. If θ is an automorphism of order n and if ζ is a primitive n–th root of unity then we can define $\mathfrak{g}_{[m]} = \{x \in \mathfrak{g} | \theta x = \zeta^m x\}$ and this is a $\mathbb{Z}/n\mathbb{Z}$ grading.

2.3 Relationship with groups 1. $\mathbb{Z}$–grades.

We now consider the case when G is a connected reductive algebraic group over $\mathbb{C}$ and $\mathfrak{g} = Lie(G)$ and $\mathfrak{g}$ is $\mathbb{Z}$–graded

$$\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_j$$

then let $\phi : \mathbb{C}^\times \to Aut(\mathfrak{g})$ be the algebraic homomorphism

$$\phi(z)v = z^j v, v \in \mathfrak{g}_j.$$

Let $\mathfrak{g} = \mathfrak{c} \oplus [\mathfrak{g}, \mathfrak{g}]$ with $\mathfrak{c}$ the center of $\mathfrak{g}$ and $[\mathfrak{g}, \mathfrak{g}]$ the commutator algebra which is semi-simple. We observe that

$$\phi(\mathbb{C}^\times)\mathfrak{c} = \mathfrak{c}$$

and

$$\phi(\mathbb{C}^\times)[\mathfrak{g}, \mathfrak{g}] \subset [\mathfrak{g}, \mathfrak{g}].$$

Let C be the center of G and let G' be the derived group then $G = CG'$. G' is connected and $Lie(G') = [\mathfrak{g}, \mathfrak{g}]$ inherits a grade from $\mathfrak{g}$. $Ad(G) = Ad(G')$ and since all of the group actions that Vinberg studies are by subgroups of the adjoint group of the Lie algebra we will emphasize G' and $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$.

For the rest of this subsection we will assume that $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$. The linear map $D : \mathfrak{g} \to \mathfrak{g}$ given by

$$D_{|\mathfrak{g}_j} = jI$$

defines a derivation of $\mathfrak{g}$. All derivaions of $\mathfrak{g}$ are inner so $D = ad(x)$ with $x \in \mathfrak{g}$. We have proved

Lemma 4 *If G is a connected semisimple algebraic group over $\mathbb{C}$ with a $\mathbb{Z}$–graded Lie algebra then there exists a semisimple element (i.e. diagonalizable) $x \in Lie(G)$ such that $ad(x)$ has integral eigenvalues and the eigenspaces of $ad(x)$ give the grade.*

2.4 Relationship with groups 2. $\mathbb{Z}/n\mathbb{Z}$–grades.

We now assume that G is a connected reductive algebraic group with $\mathfrak{g} = Lie(G)$ having a $\mathbb{Z}/n\mathbb{Z}$–grade,

$$\mathfrak{g} = \bigoplus_{[j] \in \mathbb{Z}/n\mathbb{Z}} \mathfrak{g}_{[j]}.$$

Let ζ be a primitive n–th root of unity and

$$\theta_{|\mathfrak{g}_{[j]}} = \zeta^j I.$$

As above

$$\mathfrak{g} = \mathfrak{c} \oplus [\mathfrak{g}, \mathfrak{g}]$$

and $\theta\mathfrak{c} = \mathfrak{c}$ and $\theta[\mathfrak{g}, \mathfrak{g}] = [\mathfrak{g}, \mathfrak{g}]$. The same non-interaction between the grades of the center and the derived algebras prevails.

Assumeing that G is semi-simple (i.e. $\mathfrak{c} = 0$) then $Ad : G \to Ad(G)$ is a covering homomorphism and $Ad(G)$ is the identity component of $Aut(G)$. We have

Lemma 5 *Let G be a connected semi-simple Lie group with a $\mathbb{Z}/n\mathbb{Z}$–graded Lie algebra $\mathfrak{g} = Lie(G)$. If θ is as above and $\tau(g) = \theta g \theta^{-1}$ for $g \in Ad(G)$ then identifying $\mathfrak{g}$ with $ad(\mathfrak{g})$, $d\tau = \theta$.*

2.5 The affine action associated with a grade.

Let G be a connected reductive algebraic group with $\mathfrak{g} = Lie(G)$ having a $\mathbb{Z}$ or a $\mathbb{Z}/m\mathbb{Z}$ grade with $m \in \mathbb{Z}_{>0}$. So we have a graded decomposition

$$\mathfrak{g} = \bigoplus_{s \in S} \mathfrak{g}_s$$

with $S = \mathbb{Z}$ or $\mathbb{Z}/m\mathbb{Z}$. In both cases we will write $[j]$ for the corresponding element of S. Let $\mathfrak{h}$ denote the Lie algebra $\mathfrak{g}_{[0]}$ and let $V = \mathfrak{g}_{[1]}$. Let H be the connected Lie subgroup of G with Lie algebra $\mathfrak{h}$. Then the pair $(Ad(H)_{|V}, V)$ of a group and module for the group is what we will call the affine action associated with the grade.

Lemma 6 *The group $Ad(H)_{|V}$ is a connected reductive algebraic subroup of $GL(V)$ that depends on $\mathfrak{g}$ and not the choice of G.*

Proof. We note that $V = \mathfrak{c} \cap \mathfrak{g}_{[1]} \oplus [\mathfrak{g}, \mathfrak{g}] \cap \mathfrak{g}_{[1]}$. Relative to this decomposition the elements of H act trivially on the first summand and preserve the second. We may therefore assume that $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$. If $S = \mathbb{Z}$ let $x \in \mathfrak{g}$ be such that adx gives the grade. If $S = \mathbb{Z}/m\mathbb{Z}$ then let θ be as in the previous subsection. In the first case we note that H is the identity component of $\{h \in H | Ad(h)x = x\}$ and in the second H is the identity component of

$\{h \in H | Ad(h)\theta = \theta Ad(h)\}$ in bothe cases this defines Z–closed subgroup of G and since x and θ are semi-simple H is reductive. In both cases $Ad(H)_{|V}$ is the connected subgroup of $GL(V)$ with lie algebra $\{ady | y \in \mathfrak{h}\}$ so it depends only on $\mathfrak{g}$. ■

This leads us to the main objects of our study. We use the term Z-closed for Zariski closed.

Definition 7 *A Vinberg pair will be a pair (L, V) with L a connected, reductive Z-closed subgroup of $GL(V)$ such that there exists a graded reductive Lie algebra*

$$\mathfrak{g} = \bigoplus_{s \in S} \mathfrak{g}_s$$

with $S = \mathbb{Z}$ or $\mathbb{Z}/m\mathbb{Z}$ such that $V = \mathfrak{g}_{[1]}$ and $L = Ad(H)_{|V}$ with H as in the previous Lemma.

Definition 8 *A direct sum of Vinberg pairs is a pair (L, V) with $L = L_1 \times \cdots \times L_r$ and $V = V_1 \oplus V_2 \oplus ... \oplus V_r$ with (L_i, V_i) a Vinberg pair and L acts by the block diagonal action.*

Definition 9 *A morphism of direct sums of Vinberg pairs (L, V) and (L', V') is a pair, (ϕ, T) of an algebraic group homomorphism $\phi : L \to L'$ and a linear map $T : V \to V'$ such that $Tgv = \phi(g)Tv$ for $g \in L$ and $v \in V$.*

Exercise. When is a direct sum of Vinberg pairs isomorphic with a Vinberg pair?

2.6 Decomposition into simple constituents.

In the last section we showed how one can assign a reductive algebraic group, H, and an algebraic H–module V to a reductive graded Lie algebra. In this section we will describe a decomposition of into what we will call simple constituents.

Let G and $\mathfrak{g}$ be as in the previous section. Let

$$\mathfrak{g} = \mathfrak{c} \oplus \bigoplus_{j=1}^{n} \mathfrak{g}^{(j)}$$

be a decomposition of $\mathfrak{g}$ into the direct sum of its center, $\mathfrak{c}$, and its simple ideals

$$\mathfrak{g}^{(1)}, \mathfrak{g}^{(2)}, ..., \mathfrak{g}^{(n)}.$$

If the grade on $\mathfrak{g}$ is a $\mathbb{Z}$–grade then we have seen that each of the summands inherits the grade. So in this case a $\mathbb{Z}$–graded simple constituent will just be a $\mathbb{Z}$–graded simple Lie algebra. We have

Lemma 10 *Notation as above. If $\mathfrak{g}$ is $\mathbb{Z}$–graded then $\mathfrak{g} = \mathfrak{c} \oplus \bigoplus_{j=1}^{n} \mathfrak{g}^{(j)}$ is a decomposition of $\mathfrak{g}$ into a direct sum of $\mathbb{Z}$–graded Lie algebras. Furthermore the corresponding affine group action is given as follows*

$$V = \mathfrak{c}_{[1]} \oplus \mathfrak{g}_{[1]}^{(1)} \oplus ... \oplus \mathfrak{g}_{[1]}^{(n)} = \mathfrak{c}_{[1]} \oplus V^{(1)} \oplus ... \oplus V^{(n)}$$

and if we take $G^{(i)}$ toe be the connected subgroup of G with Lie algebra $\mathfrak{g}^{(i)}$ and $H^{(i)}$ the connected subroup of $G^{(i)}$ with Lie algebra $\mathfrak{g}_{[0]}^{(i)}$ then

$$Ad(H)_{|V} = \{I_{\mathfrak{c}_{[1]}}\} \times Ad(H^{(1)})_{|V^{(1)}} \times \cdots \times Ad(H^{(n)})_{|V^{(n)}}.$$

We now consider the case when the grade is a $\mathbb{Z}/m\mathbb{Z}$–grade. The situation is almost the same except for one complication.

In this case we define $\theta : \mathfrak{g} \to \mathfrak{g}$ by $\theta_{|\mathfrak{g}_{[j]}} = \zeta^j I$ with ζ a primitive m–the root of unity. Then $\theta\mathfrak{c} = \mathfrak{c}$ and θ permutes the simple fractors. In this case the permutation breaks up into disjoint cycles. We concider one such cycle of length d and relabel so that the cycle is

$$1 \to 2 \to ... \to d \to 1.$$

We note that θ preserves $\tilde{\mathfrak{g}} = \bigoplus_{j=1}^{d} \mathfrak{g}^{(j)}$ and we analyze its action on this space. If $x \in \bigoplus_{j=1}^{d} \mathfrak{g}^{(j)}$ then $x = \sum x_i$ and relative to this decomposition we have

$$\theta x = \sum \theta x_i$$

And $\theta : \mathfrak{g}^{(j)} \to \mathfrak{g}^{(j+1)}$ for $j = 1, ..., d-1$ and $\theta : \mathfrak{g}^{(d)} \to \mathfrak{g}^{(1)}$ is a Lie algebra isomorphism. We denote the η–eigenspace for θ by sub–η. Thus $\tilde{\mathfrak{g}}_1 = \tilde{\mathfrak{g}}_{[0]}$ consists of the elements such that $\theta x_i = x_{i+1}$ for $i = 1, ..., d-1$ and $\theta x_d = x_1$. So x can be written

$$\sum_{i=0}^{d-1} \theta^i x_1$$

with the additional condition $\theta^d x_1 = x_1$. Noting that $[\mathfrak{g}^{(i)}, \mathfrak{g}^{(j)}] = 0$ if $i \neq j$ we see that if we set $\theta^{(1)}$ equal to θ^d then the map $\phi_1 : \mathfrak{g}_1^{(1)} \to \tilde{\mathfrak{g}}_1$ given by

$$\phi_1(x_1) = \sum_{i=0}^{d-1} \theta^i x_1$$

is a Lie algebra isomorphism. Furthermore if η is an m–th root of unity and if $x \in \tilde{\mathfrak{g}}_\eta$ then we have $\theta x_i = \eta x_{i+1}$, $i = 1, ..., d-1$ and $\theta x_d = \eta x_1$ so $x_i = \eta^{-i}\theta^i x_1$ and as before $\theta^d x_1 = \eta^d x_1$. We see that the map $\phi_\eta : \mathfrak{g}^{(1)}_{\eta^d} \to \tilde{\mathfrak{g}}_\eta$ given by

$$\phi_\eta(x_1) = \sum_{i=0}^{d-1} \eta^{-i}\theta^i x_1$$

defines a linear isomorphism that it addition satisfies

$$\phi_\eta([x_1, y_1]) = [\phi_1(x_1), \phi_\eta(y_1)]$$

for $x_1 \in \mathfrak{g}^{(1)}_1$ and $y_1 \in \mathfrak{g}^{(1)}_{\eta^d}$.

We can now prove the analogue of Lemma 10.

Proposition 11 *Let (L, V) be a Vinberg pair. Then (L, V) is isomorphic with a direct sum of a vector space $(\{I\}, V_o)$ and Vinberg spaces (L_j, V_j) with corresponing graded Lie algebra simple.*

The above argument suggests the following let $\mathfrak{g}$ be a Lie algebra over $\mathbb{C}$ and let $\mathfrak{g}^{\{d\}}$ be the direct sum of d copies of $\mathfrak{g}$. Let θ be an automorphism of $\mathfrak{g}$ and define

$$\theta_{\{d\}}(x_1, ..., x_d) = (\theta x_d, x_1, ..., x_{d-1})$$

then $\theta^d_{\{d\}} = (\theta, ..., \theta)$.

Lemma 12 *The map $(x_1, ..., x_d) \longmapsto \sum \theta^{i-1} x_i$ defines an isomorphism of the graded Lie algebra $(\theta^d_{\{d\}}, (\mathfrak{g}^{(1)})^{\{d\}})$ onto $(\theta, \tilde{\mathfrak{g}})$.*

Exercise. Prove this lemma.

2.7 Maximal compact groups.

In this subsection we take G to be semi-simple, connected and such that $\mathfrak{g} = Lie(G)$ is $\mathbb{Z}/m\mathbb{Z}$–graded. We assume that there exists, τ, an automorphism of G such that $d\tau = \theta$. We assume as, we may, that G is a Z–closed subgroup of $GL(n, \mathbb{C})$ that is invariant under adjoint. We set $K = G \cap U(n)$.

We note that $\tau(K)$ is a maximal compact subgroup of G (this is true for any continuous isomorphism). Thus there exists $g \in G$ such that $g\tau(K)g^{-1} = K$. Hence, if we define $\tau'(x) = g\tau(x)g^{-1}$ then τ' is an automorphism of G and

if $\theta'(X) = g\theta(X)g^{-1}$ then θ' is an element of order m such that $\tau'(e^X) = e^{\theta' X}$. We will therefor assume that $\tau(K) = K$ and $\theta(Lie(K)) = Lie(K)$.

If $\theta^m = 1$ with $m < \infty$, we also fix the inner product

$$\langle X, Y\rangle = \frac{1}{m}\sum_{l=0}^{m-1} tr(\theta^l X(\theta^l Y)^*)$$

and the symmetric bilinear form

$$(X, Y) = \frac{1}{m}\sum_{k=0}^{m-1} tr(\theta^l X(\theta^l Y)).$$

We note that, since $X \in Lie(K)$ is skew Hermitian, if $X \neq 0$, $X \in Lie(K)$ then $(X, X) < 0$. This implies that each term in the above sum for $X = Y \neq 0$ in $Lie(K)$ is negative. Thus $(..., ...)$ is negative definite on $Lie(K)$ and therefore positive definite on $iLie(K)$. In particular, $(..., ...)$ is non-degenerate on $Lie(G)$. We also note that since $\theta Ad(g)\theta^{-1} = Ad(\tau g)$ for $g \in G$ the form $(..., ...)$ is both G and θ–invariant Similarly, we have an inner product $\langle ..., ...\rangle$ that is K and θ–invariant.

We will denote $Lie(G)$ by $\mathfrak{g}$. Let for $\zeta \in \mathbb{C}$, $\mathfrak{g}_\zeta = \{X \in \mathfrak{g} | \theta X = \zeta X\}$. Then we note that if the order of θ is $m < \infty$ then

a) $\mathfrak{g}_\zeta = 0$ if $\zeta^m \neq 1$.

b) Set $G^\tau = \{g \in G | \tau(g) = g\}$ then $Lie(G^\tau) = \mathfrak{g}_1$.

c) If θ is arbitrary but semisimple then $(\mathfrak{g}_\zeta, \mathfrak{g}_\mu) = 0$ inless $\zeta\mu = 1$ and the corresponding pairing of $\mathfrak{g}_\zeta$ with $\mathfrak{g}_{\zeta^{-1}}$ is perfect. Thus in particular, $\dim \mathfrak{g}_\zeta = \dim \mathfrak{g}_{\zeta^{-1}}$.

Definition 13 *A Vinberg triple (G, V, τ) consists of a connected semi-simple algebraic group G, as above, and automorphism, τ, of G such that $\theta = d\tau$ is semi-simple as an automorphism of $\mathfrak{g} = Lie(G)$, V is the θ–eigenspace for a nontrivial eigenvalue of θ.*

There is also a notion of a θ–space which is a pair (H, V) of a connected algebraic group H and an algebraic H–module V such that there is a Vinberg triple (G, V, τ) such that H is the identity component of G^τ. We note that if (L, V) is a Vinberg pair then if L is non-trivial (L, V) is a θ–space and if (H, V) is a θ–space with H the identity component of G^τ and (G, V, τ) is a Vinberg triple then $(Ad(H)_{|V}, V)$ is a Vinberg pair

Lemma 14 *If (H,V) is a θ–space with θ of finite order. Then there exists a Vinberg triple (G,V,τ) so that if $\mathfrak{g} = Lie(G)$ then*

$$\mathfrak{g} = \bigoplus_j \mathfrak{g}_{\zeta^j}$$

with $d\tau_{|\mathfrak{g}_{\zeta^j}} = \zeta^j I$ with ζ a primitive m–th root of unity, $V = \mathfrak{g}_\zeta$ and H is the identity component of G^τ.

Proof. By the definition of θ–space there $(\tilde{G},\tau,V)$ so that H is the identity component of $\tilde{G}^\tau$ and $V = Lie(\tilde{G})_\zeta$ with ζ is a root of unity. Let m be the order of ζ then ζ is a primitive m–th root of unity. We may assume that $\tilde{G}$ is contained in $GL(n,\mathbb{C})$ and is invariant under Hermitian adjoint. Then $\mathfrak{g} = \oplus_j Lie(\tilde{G})_{\zeta^j}$ is reductive Lie subalgebra. As above we may assume that τ normalizes $\tilde{K}$ and that there is an inner product on $Lie(\tilde{G})$ so that $d\tau$ and $Ad(\tilde{K})$ act unitarily. Let K be the connected subgroup of $\tilde{K}$ with $Lie(K) = \mathfrak{g} \cap Lie(\tilde{K})$.Since $\left(Lie(\tilde{G})_\mu\right)^* = Lie(\tilde{G})_{-\mu}$ and

$$Lie(K) = \{x \in \mathfrak{g} | x^* = -x\}$$

we have $\mathrm{span}_{\mathbb{C}}(Lie(K)) = \mathfrak{g}$. We assert that K is closed. Indeed, let U be its closure. Then $(\tau_{|U})^m = I$. Thus $\mathrm{span}_{\mathbb{C}} Lie(U) = \oplus_j Lie(\tilde{G})_{\zeta^j} \cap \mathrm{span}_{\mathbb{C}} Lie(U)$ since the right hand side is contained in $\mathfrak{g}$ and the left hand side contains $Lie(K)$ we conclude that $Lie(U) = Lie(K)$ so $K = U$ (since both groups are connected). Now set G equal to the Zariski closure of U in $\tilde{G}$. Connectedness implies that the identity component of G^τ is H. This completes the proof. ■

Definition 15 *A Vindberg space is a Vindberg triple (G,τ,V) such that if $\mathfrak{g} = Lie(G)$ then $\mathfrak{g} = \bigoplus_j \mathfrak{g}_{\zeta^j}$ with either ζ a primitive m–th root of unity or ζ is of infinite order and $V = \mathfrak{g}_\zeta$.*

2.8 A lemma of Richardson

In this section we will describe some ideas of Richardson.

Lemma 16 *Let $G \subset GL(n,\mathbb{C})$ be algebraic and assume that H is a Zariski closed connected subgroup of G. Assume that $V \subset \mathbb{C}^n$ is a H–invariant subspace such that if $v \in V$ then $Lie(G)v \cap V = Lie(H)v$. If $x \in V$ then $Gx \cap V$ is a disjoint union of a finite number of orbits of H all of which are of the same dimension and are open in $Gx \cap V$*

Let $u \in Gx \cap V$. Then

$$T_u(Gx \cap V) \subset Lie(G)u \cap V = Lie(H)u \cap V.$$

Also, since

$$Hu \subset Gu \cap V = Gx \cap V$$

we see that $\dim(Gx \cap V) = \dim(Hu)$. This implies that the orbits of H in $Gx \cap V$ all have the same dimension and are open in $Gx \cap V$. Hence they are closed and since they are irreducible they are the irreducible components of the variety $Gx \cap V$.

Lemma 17 *Let (G, V, τ) be a Vinberg space and set $H = G^\tau$. If $x \in V$ then $[\mathfrak{g}, x] \cap V = [Lie(H), x]$.*

Proof. If $y \in \mathfrak{g}$ then $y = \sum_{k=0}^{m-1} y_{\zeta^k}$ (as usual). If $[y, x] \in V$ then $[y, x] = [y, x]_\zeta = [y_1, x] \in Ad(Lie(H))x$. Since it is clear that $[\mathfrak{g}, x] \cap V \supset [Lie(H), x]$ the Lemma follows. ■

2.9 Basic properties of Vinberg spaces 1.Infinite order.

Let (G, V, τ) be a Vinberg space relative to $\theta = d\tau$, $Lie(G) = \mathfrak{g} = \oplus \mathfrak{g}_{\zeta^k}$ with ζ of infinite order (all but a finite number of the $\mathfrak{g}_{\zeta^k} = 0$) and $V = \mathfrak{g}_\zeta$. We assume that K is as in the previous section and is τ–invariant. As above $H = G^\tau$. We can think of $\mathfrak{g}$ having a $\mathbb{Z}$–grading setting $\mathfrak{g}_k = \mathfrak{g}_{\zeta^k}$.

Lemma 18 *If $x \in V$ then $ad(x)$ is nilpotent.*

Proof. If $y \in \mathfrak{g}_{\zeta^k}$ then $ad(x)^r y \in \mathfrak{g}_{\zeta^{k+r}}$. Since only a finite number of $\mathfrak{g}_{\zeta^k}$ are non-zero the result follows. ■

Note in the next lemma we won't assume that ζ is of infinite order.

Lemma 19 *If (G, V, τ) is a Vinberg space then under H the identity component of G^τ the number of nilpotent orbits in V is finite.*

Proof. We replace G with $Ad(G)$ and look upon $G \subset GL(\mathfrak{g})$ a Zariski closed subgroup. We note that G acting on $\mathfrak{g}$ has a finite number of nilpotent orbits. The result now follows from Lemmas 17 and 16. ■

We note that this result implies

Proposition 20 *If (G, V, τ) is a Vinberg space such that every element $x \in V$ is nilpotent then under H the identity component of G^τ then there are only a finite number of H-orbits and there exists $v \in V$ such that Hv is Zariski open in V.*

2.10 Basic properties of Vinberg spaces 2.Finite order

Let (G, V, τ) be a Vinberg space and assume that τ is of order m and relative to $\theta = d\tau$, $Lie(G) = \mathfrak{g} = \oplus_{k=0}^{m-1} \mathfrak{g}_{\zeta^k}$ with ζ a primitive m–th root of unity and $V = \mathfrak{g}_\zeta$. We assume that K is as in the previous section and is τ–invariant. As above $H = G^\tau$

If $x \in V$ we set $x = x_s + x_n$ equal to its Jordan decomposition in $\mathfrak{g}$. That is, x_s has the property that it is a diagonalizable element of $M_n(\mathbb{C})$(i.e. semisimple), x_n is nilpotent and $[x_s, x_n] = 0$. This decomposition is unique. We note that if $x \in \mathfrak{g}$ is semisimple then θx is also semisimple similarly if x is nilpotent θx is nilpotent. Thus $\theta(x_s) = \theta(x)_s$ and $\theta(x_n) = \theta(x)_n$. Since, $x \in \mathfrak{g}_\zeta$ we have $\theta x = \zeta x$ so $x_s, x_n \in V$.

Theorem 21 *If $x \in V$ then Hx is closed if and only if $x = x_s$.*

Proof. We first prove that if $x \in V$ then $\overline{Hx}$ contains x_s. If $x_n = 0$ there is nothing to prove. Otherwise, set $e = x_n$ and let $f, h \in \mathfrak{g}_{x_s}$(that is commute with x_s) be such that

$$[e, f] = h, [h, f] = -2f, [h, e] = 2e$$

Then $h = \sum_{k=0}^{m-1} h_{\zeta^k}$ with $\theta h_{\zeta^k} = \zeta^k h_{\zeta^k}$. Thus $\theta[h, e] = 2\zeta x$ on the one hand and

$$\theta[h, e] = \sum_{k=0}^{m-1} \zeta^{k+1} [h_{\zeta^k}, e].$$

This implies that $[h_{\zeta^k}, e] = 0$ unless $z^k = 1$. Similarly, $[h_{\zeta^k}, x_s] = 0$ for all k. Thus there exists $u = h_1 \in Lie(H)$ such that $[u, x_s] = 0$ and $[u, x_n] = 2x_n$. Hence

$$Ad(e^{-tu})x = x_s + e^{-2t} x_n.$$

This implies our assertion and therefore proves that if x isn't semisimple in $\mathfrak{g}$ then Hx isn't closed. So if Hx is closed then x is semisimple in $\mathfrak{g}$.

We now show that if x is semisimple then Hx is closed. We note that $\overline{Hx}$ contains a closed orbit, Hu with $u \in V$. We note that $u \in \overline{Gx} = Gx$ since x is semisimple in $\mathfrak{g}$. Lemmas 17 and 16. above prove that $\dim Ad(H)x = \dim Ad(H)u$. Since $Ad(H)u$ is closed, irreducible and contained in $\overline{Ad(H)x}$ which is irreducible we see that $Ad(H)u = Ad(H)x$. ■

3 The structure of a Vinberg pair

In this section we delve into the more delicate aspects of Vinbergs results.

3.1 Cartan Subspaces

Definition 22 *Let (G, V, τ) be a Vinberg space. A Cartan subspace of V is a subspace that consists of semi-simple commuting elements and is maximal subject to this property. The rank of V is the maximum of the dimensions of Cartan subspaces.*

Remark 23 *The rank of (G, V, τ) is 0 if and only if every element of V is nilpotent.*

For the rest of this section we will assume that (G, V, τ) is a Vinberg space with $\theta = d\tau$ of order m. We will also asume that G is connected.

Fix $\mathfrak{a}$ a Cartan subspace of V. Let $T \subset G$ be the intersection of all Zariski closed sugroups, U, of G such that $Lie(U) \supset \mathfrak{a}$.

Lemma 24 *T is an algebraic torus that is contained in the identity component of the center of the centralizer of $\mathfrak{a}$ in G.*

Proof. Consider the Z–closure, T_1, of $\exp(\mathfrak{a})$. Then T_1 is abelian and it is contained in every Z–closed subgroup that whose Lie algebra contains $\mathfrak{a}$. Thus since the Lie algebra of a Lie group is the same as the Lie algebra of its identity component we see that $T_1 = T$ and T is connected and abeleian. Let G_1 be the centralzer of $\mathfrak{a}$ in G then G_1 is reductive furthermore the center C of G_1 has the property that $Lie(C) \supset \mathfrak{a}$. Hence T is contained in the identity component of the center of G_1. Hence, T consists of semi-simple elements and is connected so it is a torus. ■

We will set $T = T_{\mathfrak{a}}$.

Proposition 25 $\dim T_{\mathfrak{a}} = \varphi(m) \dim \mathfrak{a}$ *with $\varphi(m)$ the Euler Totient function (the number of $0 < j < m$ with $\gcd(j, m) = 1$).*

Proof. Set $T = T_{\mathfrak{a}}$. Then $\tau : T \to T$ is an algebraic automorphism. The character group, $\hat{T}$, of T is a free abelian group of rank equal to $\dim T$. Thus if we identify a character with its differential we find that $Lie(T)^*$ has a basis $\lambda_1, ..., \lambda_d$ such that $\theta = d\tau$ has an integral matrix. Let $e_1, ..., e_d$ be the

dual basis to $\lambda_1, ..., \lambda_d$ then $\theta e_i = \sum a_{ji} e_j$ with $a_{ij} \in \mathbb{Z}$. The characteristic polynomial, $f(x)$, of $\theta_{|Lie(T)}$ is

$$f(x) = \sum_{k=0}^{d} c_k x^k$$

with $c_j \in \mathbb{Z}$. But since $\theta^m = 1$ we have a factorization over $\mathbb{Q}[\zeta]$

$$f(x) = \prod_{k=0}^{m-1} (x - \zeta^k)^{d_k}.$$

By definition of $T_{\mathfrak{a}}$ and the maximality of $\mathfrak{a}$, $d_1 = \dim \mathfrak{a}$. Let for $0 < j < m$ and $\gcd(j, m) = 1$, $\sigma : \mathbb{Q}[\zeta] \to \mathbb{Q}[\zeta]$ be the element of the Galois group $\mathbb{Q}[\zeta]$ over $\mathbb{Q}$ defined by $\sigma(\zeta) = \zeta^j$. Thus

$$f(\sigma(x)) = \sum_{k=0}^{d} c_k \sigma(x)^k = \sigma(\sum_{k=0}^{d} c_k x^k)$$

also

$$\sigma(f(x)) = \prod_{k=0}^{m-1} (\sigma(x) - \zeta^{kj})^{d_k}$$

and

$$f(\sigma(x)) = \prod_{k=0}^{m-1} (\sigma(x) - \zeta^k)^{d_k}.$$

This implies that $d_j = d_1$. Hence, $\dim Lie(T) \cap \mathfrak{g}_{\zeta^j} = \dim \mathfrak{a}$. This yields the lower bound $\dim T \geq \varphi(m) \dim \mathfrak{a}$.

To prove the upper bound we show that if

$$\mathfrak{b} = \sum_{\gcd(m,j)=1} Lie(T) \cap \mathfrak{g}_{\zeta^j}$$

then $\exp(\mathfrak{b})$ is Zariski closed in T. We may assume that $T \subset (\mathbb{C}^\times)^n$. We note that if h is the m–th cyclotomic polynomial that $\mathfrak{b} = \ker h(\theta_{|Lie(T)})$. Let

$$h(x) = \sum_{j=0}^{\varphi(m)-1} r_j x^j.$$

As is standard, $a_j \in \mathbb{Z}$. We consider the regular homomorphism of T to T.

$$\beta(z) = \prod_{j=0}^{\phi(m)-1} \tau(z)^{r_j}.$$

If $z = \exp(u)$ with $u \in \mathfrak{b}$ then

$$\beta(z) = \exp(\sum_{j=0}^{\varphi(m)-1} r_j \theta^j u) = 1.$$

Thus $\ker d\beta$ is equal to $\mathfrak{b}$. Hence the identity compoent of $\ker \beta$ is an algebraic subtorus of T whose Lie algebra contains $\mathfrak{a}$. This implies that $T = \exp \mathfrak{b}$ and so $\dim T \leq \varphi(m) \dim \mathfrak{a}$. ■

3.2 Restricted roots

We continue with the notation of the previous section.

Let $\mathfrak{a}$ be a Cartan subspace of V and let $\Sigma(\mathfrak{a})$ be the set elements $\lambda \in \mathfrak{a}^*$, $\lambda \neq 0$ such that there exists $x \in \mathfrak{g}$ such that $x \neq 0$ and $[h, x] = \lambda(h)x$ for all $h \in \mathfrak{a}$. Then $\Sigma(\mathfrak{a})$ is called the *restricted root system* of the triple. If $\lambda \in \mathfrak{a}^*$ set

$$\mathfrak{g}^\lambda = \{x \in \mathfrak{g} | [h, x] = \lambda(h)x, h \in \mathfrak{a}\}.$$

Then

$$\mathfrak{g} = \mathfrak{g}^0 \bigoplus \bigoplus_{\lambda \in \Sigma(\mathfrak{a})} \mathfrak{g}^\lambda.$$

Assume that $\Sigma(\mathfrak{a}) = \emptyset$. The space $\mathfrak{a}$ consists of semi-simple elements thus $\mathfrak{a}$ is contained in the center of $\mathfrak{g}$. The maximality in the definition of a Cartan subspace implies that $\mathfrak{a}$ is the unique Cartan subspace. Also, since every semi-simple element in V is contained in some Cartan subspace we see that every semi-simple element in V is in $\mathfrak{a}$. Lemma 24 implies that $T = T_{\mathfrak{a}}$ is central. Setting $G' = G/T$ and $\mathfrak{g}' = Lie(G')$ then $(G', V/\mathfrak{a}, \tau_{|G'})$ is a Vinberg triple of rank 0. We eliminate this complication by adhering to the following definition:

Definition 26 *A Vinberg triple will be called semi-simple if G is semi-simple (that is, $Lie(G)$ has 0 center). We will abreviate this to SS Vinberg triple.*

Lemma 27 *Assume that (G, V, τ) is a semi-simple Vinberg triple. If $\mathfrak{a}$ is a non-zero Cartan subspace of V then the set $\Sigma(\mathfrak{a})$ spans the dual space of $\mathfrak{a}$.*

Proof. If $h \in \mathfrak{a}$ satisfies $\Sigma(\mathfrak{a})(h) = \{0\}$ then since h is semi-simple $[h, \mathfrak{g}] = 0$. Thus $h = 0$. ■

For the rest of this section we will assume that the rank of (G, V, τ) is semi-simple and not 0. Fix for the moment a Cartan subspace, $\mathfrak{a}$. Then an element, h, of $\mathfrak{a}$ is called regular if $\lambda(h) \neq 0$ for all $\lambda \in \Sigma(\mathfrak{a})$. We set $\mathfrak{a}'$ equal to the set of regular elements of $\mathfrak{a}$.

If $S \subset V$ is a set of commuting semi-simple elements then we set $G^S = \{g \in G | gs = s, s \in S\}$. Then G^S is a reductive alebraic group and

$$(G^S, Lie(G^S) \cap V, \tau_{|G^S})$$

is a Vinberg triple. We note the $Lie(G^S) = \{x \in \mathfrak{g} | [x, s] = 0, s \in S\} = C_{\mathfrak{g}}(S)$.

Lemma 28 *If $x \in C_{\mathfrak{g}}(\mathfrak{a})_\zeta$ then $x_s \in \mathfrak{a}$.*

Proof. We have seen that $x_s \in V$. Furthermore, if $[x, h] = 0$ then $[x_s, h] = 0$. The lemma follows. ■

Lemma 29 *Let $h \in \mathfrak{a}'$ then $C_{\mathfrak{g}}(h)_\zeta = C_{\mathfrak{g}}(\mathfrak{a})_\zeta$.*

Proof. This follows since the restricted root decomposition implies $C_{\mathfrak{g}}(\mathfrak{a}) = \mathfrak{g}^0 = C_{\mathfrak{g}}(h)$. ■

Proposition 30 *Let H be the identity component of G^τ. Set*

$$r(\mathfrak{a}) = \{x \in C_{\mathfrak{g}}(\mathfrak{a})_\zeta | x_s \in \mathfrak{a}'\}.$$

Then $Ad(H)r(\mathfrak{a})$ has non-empty Zariski interior in V.

Proof. We first prove that the map

$$\Phi : H \times r(\mathfrak{a}) \to V$$

$$(h, x) \mapsto Ad(h)x$$

has surjective differential for all $(h, x) \in H \times r(\mathfrak{a})$. Let $(..., ...)$ be the form defined in section 2.2 then both G and θ leave it invariant. This implies

that it induces a perfect pairing between $V = \mathfrak{g}_\zeta$ and $\mathfrak{g}_{\zeta^{-1}}$. We note that if $x, v \in r(\mathfrak{a})$ and $h \in H$ then if $X \in Lie(H)$ then

$$d\Phi_{h,x}(X_h, v) = Ad(h)([X, x] + v).$$

If this differential is not surjective then there would exist $u \in \mathfrak{g}_{\zeta^{-1}}$ such that

$$(u, [X, x] + v) = 0$$

for all $v \in r(\mathfrak{a})$ and $X \in Lie(H) = \mathfrak{g}_1$. this implies that

$$0 = (u, [X, x]) = ([x, u], X)$$

so $[u, x]$ is orthogonal to $\mathfrak{g}_1$. So since $[u, x] \in \mathfrak{g}_1$ this implies that $[u, x] = 0$. As before, $\ker adx \subset \ker adx_s$ which $u \in C_{\mathfrak{g}}(\mathfrak{x}_s) = C_{\mathfrak{g}}(\mathfrak{a})$. Hence $u \in C_{\mathfrak{g}}(\mathfrak{a})_{\zeta^{-1}}$ and is orthogonal to $C_{\mathfrak{g}}(\mathfrak{a})_\zeta$. Thus $u = 0$. This implies that Φ has open image in the Euclidian topology. Thus the image of Φ is Zariski dense and so has non-zero Zariski interior. ■

Corollary 31 *If $\mathfrak{a}$ and $\mathfrak{b}$ are Cartan subspaces of V then there exists $h \in H$ such that $h\mathfrak{a} = \mathfrak{b}$.*

Proof. The sets $Ad(H)r(\mathfrak{a})$ and $Ad(H)r(\mathfrak{b})$ both have non-empty Zarisky interior. This implies that they must have non-trivial intersection. Thus there is $h \in H$, $x \in r(\mathfrak{a})$ such that $Ad(h)x \in r(\mathfrak{b})$. Thus $Ad(h)x_s \in \mathfrak{b}$. So $Ad(h)\mathfrak{a} \subset \mathfrak{b}$. The maximality implies equality. ■

Corollary 32 *Let $\mathfrak{a}$ be a Cartan subspace of V. If $x \in V$ and $Ad(H)v$ is closed in V then there exists $h \in H$ such that $Ad(h)x \in \mathfrak{a}$.*

Proof. If $\mathfrak{b}$ is maximal in V subject to the conditions that pairs of elements in $\mathfrak{b}$ commute, every element in $\mathfrak{b}$ is semisimple and $x \in \mathfrak{b}$. Then $\mathfrak{b}$ is a Cartan subspace of V so the previous corollary implies that there is an $h \in H$ such that $Ad(h)\mathfrak{b} = \mathfrak{a}$. ■

Corollary 33 *Let $\mathfrak{a}$ be a Cartan subspace of V. Then $Ad(H)r(\mathfrak{a})$ is Zariski open in V.*

Proof. Let for $X \in V$

$$\det(tI - adX) = \sum_{j=0}^{p} t^j D_{p-j}(X)$$

with $p = \dim \mathfrak{g}$. Since each D_{p-j} is H invariant $D_{p-j}(X) = D_{p-j}(X_s)$. There exists $h \in H$ and $x \in \mathfrak{a}$ such that $X_s = Ad(h)x$. For $x \in \mathfrak{a}$ we have

$$\det(tI - adX) = t^{\dim \mathfrak{g}_0} \prod_{\lambda \in \Sigma(\mathfrak{a})} (t - \lambda(x))^{\dim \mathfrak{g}^\lambda}.$$

Set $u = \dim \mathfrak{g}^0$. Thus if $x \in \mathfrak{a}$ then $D_{p-j} = 0$ for $j < u$ and

$$D_{p-u}(x) = (-1)^{p-u} \prod_{\lambda \in \Sigma(\mathfrak{a})} \lambda(x)^{\dim \mathfrak{g}^\lambda}.$$

This implies that $Ad(H)r(\mathfrak{a}) = \{x \in V | D_{p-u}(x) \neq 0\}$. ■

3.3 The Weyl group

If G is a Lie group, H is a closed subgroup and $\mathfrak{a}$ is a subspace of $Lie(G)$ then the normalizer of $\mathfrak{a}$ in H is the subgroup

$$N_H(\mathfrak{a}) = \{g \in H | Ad(g)\mathfrak{a} \subset \mathfrak{a}\}$$

and the centralizer of $\mathfrak{a}$ is the subgroup

$$C_H(\mathfrak{a}) = \{g \in H | Ad(g)x = x, x \in \mathfrak{a}\}.$$

The Weyl group of $\mathfrak{a}$ is the group

$$W_H(\mathfrak{a}) = N_H(\mathfrak{a})/C_H(\mathfrak{a})$$

which we will think of as a group of linear maps of $\mathfrak{a}$ to $\mathfrak{a}$.

Lemma 34 *If* $\mathrm{ad}x$ *is diagonalizable for every element in* $\mathfrak{a}$ *and* $[x, y] = 0, x, y \in \mathfrak{a}$ *then* $Lie(N_H(\mathfrak{a})) = Lie(C_H(\mathfrak{a}))$.

Proof. We note that

$$Lie(N_H(\mathfrak{a})) = \{x \in Lie(H) | \mathrm{ad}(x)\mathfrak{a} \subset \mathfrak{a}\}$$

and

$$Lie(C_H(\mathfrak{a})) = \{x \in Lie(H) | \mathrm{ad}(x)\mathfrak{a} = 0\}.$$

If $x \in Lie(N_H(\mathfrak{a}))$ and $y \in \mathfrak{a}$ then $[x, y] \in \mathfrak{a}$ so $(\mathrm{ad}y)^2 x = 0$. This implies that $[y, x] = 0$ since $\mathrm{ad}y$ is diagonalizable. Hence $y \in Lie(C_H(\mathfrak{a}))$. ■

Corollary 35 *If $\mathfrak{a}$ satisfies the conditions of the previous lemma and if G is a linear algebraic group and H is a Z–closed subgroup then $W_H(\mathfrak{a})$ is finite.*

Proof. The previous lemma implies that $C_H(\mathfrak{a})$ contains the identity component of $N_H(\mathfrak{a})$ since an algebraic group has a finite number of connected components the result follows. ■

We now return to the situation of the previous section.

Corollary 36 *Let $\mathfrak{a}$ be a Cartan subspace of V then $W_H(\mathfrak{a})$ is a finite group which we call the Weyl group of the Vinberg space.*

We note that since any two Cartan spaces of V are conjugate by H the Weyl group is, up to isomorphism, independent of the choice of $\mathfrak{a}$.

Proposition 37 *If $x, y \in \mathfrak{a}$, a Cartan subspace of V then $Ad(H)x = Ad(H)y$ if and only if $W_H(\mathfrak{a})x = y$.*

Proof. We assume $x, y \in \mathfrak{a}$ and that there exists $h \in H$ with $hx = y$. We consider the Vinberg triple $(C_G(y), Lie(C_G(y)) \cap V, \tau_{|C_G(y)})$. Then $\mathfrak{a}$ and $Ad(h)\mathfrak{a}$ are Cartan subspaces of $Lie(C_G(y)) \cap V$. Hence there exists u in the identity component of $C_G(y)^\tau$ such that $Ad(u)Ad(h)\mathfrak{a} = \mathfrak{a}$. We have $Ad(uh)x = Ad(u)y = y$. Thus $uh \in N_H(\mathfrak{a})$ and $Ad(uh)x = y$. ■

We now come to Vinberg's generalization of the Chevalley restriction theorem. Our proof is a bit simpler than the original.

Theorem 38 *Assume that G is semi-simple. The restriction map $\mathrm{res}_{V/\mathfrak{a}} : \mathcal{O}(V)^H \to \mathcal{O}(\mathfrak{a})^{W_H}$ is an isomorphism of algebras.*

Proof. Set $W = W_H(\mathfrak{a})$. We first observe

1. If $x, y \in \mathfrak{a}$ and $Wx \cap Wy = \emptyset$ then there exists $f \in \mathcal{O}(V)^H$ with $f(x) = 0$ and $f(y) = 1$.

Indeed, if $x, y \in \mathfrak{a}$ then $Ad(H)x$ and $Ad(H)y$ are Z-closed in $\mathfrak{a}$. The previous proposition implies that $Ad(H)x \cap Ad(H)y = \emptyset$. Theorem 36 in [GIT] asserts the existance of f.

Set $B = \mathrm{res}_{V/\mathfrak{a}}(\mathcal{O}(V)^H)$ and let $q(B)$ be the quotient field of B. Let F be the field of rational functions on $\mathfrak{a}$. Then F is the quotent field of $\mathcal{O}(\mathfrak{a})$. We assert that

2. F is a normal extension of $q(B)$.

Consider,

$$\det(tI - adX) = \sum_{j=0}^{p} t^j D_{p-j}(X)$$

for $X \in V$ (see the previous sub-section) then $D_{p-j} \in \mathcal{O}(V)^H$ so $h(t) = \sum_{j=0}^{p} t^j \mathrm{res}_{V/\mathfrak{a}} D_{p-j}(X)$ is in $q(B)[t]$. The roots of this polynomial are 0 and the elements of $\Sigma(\mathfrak{a})$. Since the span of $\Sigma(\mathfrak{a})$ is $\mathfrak{a}^*$ we see that F is the splitting field of $h(t)$ and thus a normal extension of $q(B)$. This implies that

$$q(B) = \{f \in F | \sigma f = f, \sigma \in Gal(F/q(B))\}.$$

Also,

3. If $\sigma \in Gal(F/q(B))$ then $\sigma(\mathfrak{a}^*) = \mathfrak{a}^*$ hence $\sigma\mathcal{O}(\mathfrak{a}) = \mathcal{O}(\mathfrak{a})$.

Let $U = \{\sigma \in Aut(\mathcal{O}(\mathfrak{a})) | \sigma_{|B} = I\}$.Then $U = Gal(F/q(B))_{|\mathcal{O}(\mathfrak{a})}$. We now come to the crux of the argument. Let $\sigma \in U$ then we set for $x \in \mathfrak{a}$, $f \in \mathcal{O}(\mathfrak{a})$

$$\delta_{\sigma,x}(f) = \sigma f(x).$$

This defines a homomorphism of $\mathcal{O}(\mathfrak{a})$ to $\mathbb{C}$. The Nullstellensatz implies that there exists $x_1 \in \mathfrak{a}$ such that $\delta_{\sigma,x}(f) = f(x_1)$ for all $f \in \mathcal{O}(\mathfrak{a})$. By the definition of U we see that $f(x_1) = f(x)$ for all $f \in B$. 1. implies that there exists $s \in W$ so that $x_1 = sx$. Hence if $f \in \mathcal{O}(\mathfrak{a})^W$ then $\sigma f = f$. This implies that $\mathcal{O}(\mathfrak{a})^W \subset B$. Since the converse is obvious the result follows. ■

3.4 The GIT quotient

Let (H, V) be a Vinberg pair corresponding to a graded Lie algebra, $\mathfrak{g} = \oplus_{k=0}^{m-1} \mathfrak{g}_{\zeta^k}$ with ζ a primitive m–th root of unity. Then $\mathcal{O}(V)^H$ is a Noetherian algebra and the variety $\mathrm{spec}_{\max}\mathcal{O}(V)^H$ is an affine variety denoted $V//H$

(the categoric or GIT quotient). Since $\mathcal{O}(V)^H$ is a subalgebra of $\mathcal{O}(V)$ we have a morphism $\Phi : V \to V//H$ Let $\mathfrak{a}$ be a Cartan subspace of V and let $l = \dim \mathfrak{a} = \mathrm{rank}(\mathfrak{g})$.

Theorem 39 *Each fiber of Φ is connected, of codimension l and has a finite number of H–orbits. In paricular, each fiber has a unique closed orbit and a unique open orbit.*

Proof. We note that the fiber $\Phi^{-1}(\Phi(0))$ is the set of nilpotent elements in V which decomposes into a finite number of H–orbits (Theorem 19). We note that $\Phi(x) = \Phi(y)$ if and only if $Hx_s = Hy_s$. Indeed, consider the set $F_x = \Phi^{-1}(\Phi(x))$. We note that if $y \in F_x$ then $f(x) = f(y)$ for all $f \in \mathcal{O}(V)^H$. This implies that $f(x_s) = f(y_s)$ for all $f \in \mathcal{O}(V)^H$. This implies that $Hx_s = Hy_s$. We also have $x = x_s + y_s$ with $[x_s, y_s] = 0$. Thus the set of elements in F_x is precisely the collection of elements in V of the form $h(x_s + z)$ with z nilpotent and in $C_{\mathfrak{g}}(x_s) \cap V$. Indeed, if $y \in F_x$ we have seen that $y_s = hx_s$ for some $h \in H$. Thus

$$h^{-1}y = x_s + h^{-1}y_s = x_s + z$$

implying the assertion. But $(H_{x_s}, C_{\mathfrak{g}}(x_s) \cap V)$ is a Vindberg pair. This implies that the set of such z breaks up into a finite number of H_{x_s} orbits.

Let V'' be the set of points $v \in V$ such that $\dim Ad(H)v$ is of maximal dimension. Then V'' is Zariski open and dense in V. This implies that the maximal dimension is the generic dimension. Since each fibre of Φ contains an open orbit and the generic dimension of the fiber of Φ is the minimal dimension (see, for example, Shafarevich, Basic Algebraic Geometry, p.60, Theorem 1.3.7) we must have all fibers have the same dimension. We note that the dimension of $V//H$ is the same as that of $\mathfrak{a}/W$ which is l. Thus the minimal dimension of a fiber is $\dim V - l$. ■

3.5 Complex reflections

We assume that (G, V, τ) is a semi-simple Vinberg Space with $\theta = d\tau$ and $\theta^m = 1$. Let $\mathfrak{a}$ be a Cartan subspace of V and let $\Sigma(\mathfrak{a})$ be the root system of $\mathfrak{a}$. Let $\lambda \in \Sigma(\mathfrak{a})$ and set $\mathfrak{g}^{[\lambda]}$ equal to the

$$\mathfrak{g}^0 \oplus \bigoplus_{\mu \in \Sigma(\mathfrak{a}) \cap \mathbb{C}\lambda} \mathfrak{g}^\mu.$$

We now collect some properties of $\mathfrak{g}^{[\lambda]}$.

Lemma 40 *The subalgebra $\mathfrak{g}^{[\lambda]}$ is invariant under θ. In fact, $\theta\mathfrak{g}^{\mu} = \mathfrak{g}^{\zeta^{-1}\mu}$.*

Proof. The last assertion implies the first. Let $y \in \mathfrak{g}^{\mu}$ and let $x \in \mathfrak{a}$. Then

$$[x, \theta y] = \theta[\theta^{-1}x, y] = \zeta^{-1}\mu(x)\theta y.$$

■

Lemma 41 *$\mathfrak{g}^{[\lambda]}$ is reductive.*

Proof. Let B denote the Killing form of $\mathfrak{g}$. Then we have $B(\mathfrak{g}^{\mu}, \mathfrak{g}^{\nu}) = 0$ unless, $\mu + \nu = 0$. This implies that B is nondegenerate on $\mathfrak{g}^{[\lambda]}$. ■

Lemma 42 *Let $y \in \mathfrak{g}^{\mu}$ and write $y = \sum_{j=0}^{m-1} y_{\zeta^j}$ relative to the grade on $\mathfrak{g}$ correspondng to θ then if $x \in \mathfrak{a}$ and $\mu(x) \neq 0$ then*

1. $(\frac{ad(x)}{\mu(x)})^m y_1 = y_1$.
2. $y_{\zeta^j} = (\frac{ad(x)}{\mu(x)})^j y_1, j = 0, ..., m-1$.

Proof. $\lambda(x)y = \sum_j ad(x)y_{\zeta^j}$. Since $x \in \mathfrak{g}_{\zeta}$, $ad(x)y_{\zeta^j} = \lambda(x)y_{\zeta^{j+1}}$. This implies the result. ■

We consider, $[\mathfrak{g}^{[\lambda]}, \mathfrak{g}^{[\lambda]}]$ which is a semi-simple Lie algebra that is $\mathbb{Z}/m\mathbb{Z}$ graded. We also note that $\mathfrak{a} \cap [\mathfrak{g}^{[\lambda]}, \mathfrak{g}^{[\lambda]}] \neq 0$. Also, the restriced root system of $\mathfrak{g}^{[\lambda]}$ on $\mathfrak{a}$ consists of multiples of λ. Thus we see that Lemma 27 imples that

$$\dim\left([\mathfrak{g}^{[\lambda]}, \mathfrak{g}^{[\lambda]}] \cap \mathfrak{a}\right) = 1.$$

We have proved

Lemma 43 *There exists a unique element of $H_{\lambda} \in [\mathfrak{g}^{[\lambda]}, \mathfrak{g}^{[\lambda]}] \cap \mathfrak{a}$ such that $\lambda(H_{\lambda}) = 1$.*

Proposition 44 *Let (G, V, τ) be a rank one semi-simple Vinberg space then its Weyl group is a non-trivial cyclic group.*

Proof. Let (L, V) be the corresponing Vinberg pair. Lemma 11 implies that V is isomorphic with a direct sum of simple Vinberg pairs. Since the rank is one only one of the pairs has positive rank, which is one. Thus we can assume that $\mathfrak{g} = Lie(G)$ is simple and that $\theta = d\tau$ has order m. Let $\mathfrak{a}$ be a Cartan subspace of V. Then since the rank is 1, $\Sigma(\mathfrak{a}) \subset \mathbb{C}\lambda$ with λ a restricted root. This implies that the Weyl group is isomorphic with a subgroup of $\mathbb{C}^{\times}$. Thus

it is a cyclic group (see the appendix to this subsection). Assume that the group is trivial. We will derive a contradiction. Since $\mathcal{O}(\mathfrak{a})^W = \mathcal{O}(\mathfrak{a})$ we se that $\mathcal{O}(V)^H = \mathbb{C}[u]$ with $u \in V^*$. Let $H_\lambda \in \mathfrak{a}$ be defined by $\lambda(H_\lambda) = 1$. Since H_λ is semi-simple and $\mathfrak{a} = \mathbb{C}H_\lambda$ we may assume that $u(H_\lambda) = 1$.We also note that $\ker u$ is H-invariant thus complete reducibility implies that

$$V = \mathbb{C}v \oplus \ker u$$

with $Ad(H)v \subset \mathbb{C}v$. We also note that since u generates the H–invariants $\ker u$ is precisely the set of nilpotent elements in V. Let $Ad(h)v = \chi(h)v$ be the action of H on $\mathbb{C}v$. Let $v = v_s + v_n$ be its Jordan decomposition. If $h \in H$,

$$Ad(h)v_s = (Ad(h)v)_s = \chi(h)v_s.$$

Since $u(v) \neq 0$, $v_s \neq 0$. So we may assume $v = v_s$. There exists $h \in H$ such that $Ad(h)v = cH_\lambda$ thus

$$\chi(h)v = cH_\lambda.$$

We therefore see that we may assme that $v = H_\lambda$.Since the Weyl group is trivial we see that χ is the trivial character. We are finally ready for the contradiction. Let $y \in \mathfrak{g}^\lambda$ and let $y = \sum_{k=0}^{m-1} y_{\zeta^k}$($\zeta$ a primitive m–th root of unity) relative to the grade. Lemma 42 implies that

$$y_{\zeta^k} = ad(H_\lambda)^k y_1$$

and $ad(H_\lambda)^m y_1 = y_1$. But $Ad(H)$ fixes H_λ. Hence $y = 0$. This is the desired contradiction. ■

Definition 45 *If X is a complex vector space then a complex reflection is a linear transformation $s : X \to X$ such that $\ker(s - I)$ has codimension 1 and it has a non-trivial root of unity as an eigenvalue.*

Proposition 46 *Let (G, V, τ) be a semi-simple Vinberg space and let $\mathfrak{a}$ be a Cartan subspace of V with $\Sigma(\mathfrak{a})$ the corresponding root system and let H be the identity component of G^τ. Let $\lambda \in \Sigma(\mathfrak{a})$ then there exists a complex reflection $s_\lambda \in W_H(\mathfrak{a}) = W$ such that $\ker(s_\lambda - I) = \ker \lambda$ and the order of s_λ is the order of the Weyl group of $\mathfrak{g}^{[\lambda]}$.*

Proof. We note that if $y \in \mathfrak{g}^{[\lambda]}$ and if $x \in \mathfrak{a}$ and $\lambda(x) = 0$ then $[x, y] = 0$. Let $G^{[\lambda]}$ be the connected group correseponding to $\mathfrak{g}^{[\lambda]}$ and let $H^{[\lambda]}$ be the

identity component of $H \cap G^{[\lambda]}$. Then we know that $W_{H^{[\lambda]}}([\mathfrak{g}^{[\lambda]}, \mathfrak{g}^{[\lambda]}] \cap \mathfrak{a})$ is a cyclic group. Let s be a generator and let l be its order. Let $h \in H^{[\lambda]} \subset H$ be a representative. Then $Ad(h)H_\lambda = \mu H_\lambda$ with μ a primative l–th root of unity. Also, $Ad(h)_{|\ker\lambda} = I$. Thus $Ad(h)_{|\mathfrak{a}}$ is a complex reflection of order l. ■

Lemma 47 *We continue the notation in the previous proposition. If $\lambda \in \Sigma(\mathfrak{a})$ and $x \in \mathfrak{a}$ then*

$$s_\lambda(x) = x + (\mu - 1)\lambda(x)H_\lambda$$

with μ a primative l–th root of unity.
Proof. $\mathfrak{a} = \mathbb{C}H_\lambda \oplus \ker\lambda$. *If $x \in \mathfrak{a}$ then $x = \lambda(x)H_\lambda + (x - \lambda(x)H_\lambda)$ is its ecpression relative to the direct sum decomposition. Thus*

$$s_\lambda(x) = \mu\lambda(x)H_\lambda + x - \lambda(x)H_\lambda.$$

■

We note that the reflection s_λ depends on the choice of primitive l–root of unity but otherwise is the same for any multiple of λ that is a root.

Proposition 48 *We maintain the notation of the Proposition above. Let $\tilde{W}$ be the subgroup of W generated by the reflections s_λ for $\lambda \in \Sigma(\mathfrak{a})$. Then $\tilde{W}$ is a normal subgroup of W. If $\mathfrak{g}$ is simple then $\tilde{W}$ acts irreducibly on $\mathfrak{a}$.*

Proof. If $s \in W$ and $\lambda \in \Sigma(\mathfrak{a})$ then $s\lambda = \lambda \circ s^{-1} \in \Sigma(\mathfrak{a})$. This an the previous Lemma implies that $\tilde{W}$ is normal. Suppose that $\mathfrak{g}$ is simple of $\tilde{W}$ is and $\mathfrak{a} = A \oplus B$ with $\tilde{W}A = A$ and $\tilde{W}B = B$. If $a \in A$ then $s_\lambda(a) = (l-1)\lambda(a)H_\lambda + a$. Thus if $\lambda(a) \neq 0$ then $H_\lambda \in A$. This implies that if $\lambda_{|A} \neq 0$ then $\lambda_{|B} = 0$ and vice-versa. Set $\Sigma_A = \{\lambda \in \Sigma(\mathfrak{a}) | \lambda_{|A} \neq 0\}$ and $\Sigma_B = \{\lambda \in \Sigma(\mathfrak{a}) | \lambda_{|B} \neq 0\}$. Then $\Sigma(\mathfrak{a}) = \Sigma_A \cup \Sigma_B$ a disjoint union (since $\Sigma(\mathfrak{a})$ spans the dual space of $\mathfrak{a}$). We also note that the above observation impleis that if $\lambda \in \Sigma_A$ and $\mu \in \Sigma_B$ then $c\lambda + d\mu \in \Sigma(\mathfrak{a})$ only if $cd = 0$. Let

$$Z_A = \sum_{\lambda \in \Sigma_A} \mathfrak{g}^\lambda, Z_B = \sum_{\lambda \in \Sigma_B} \mathfrak{g}^\lambda$$

and let $\mathfrak{g}_A$ (resp. $\mathfrak{g}_B$) be the Lie subalgebra of $\mathfrak{g}$ generated by Z_A (resp. Z_B). Noting that $[Z_A, Z_B] = 0$ we see that $\mathfrak{g}_A$ and $\mathfrak{g}_B$ are ideals in $\mathfrak{g}$. Thus one of them must be 0. Hence A or B is 0. ■

We finally come to the crux of the matter.

3.5.1 Appendix

The purpose of this appendix is to prove two technical Lemmas needed for the subsection above and to recall a standard result with a reference. The first uses an argument that was posted by Andrea Petracci on the internet.

Lemma 49 *If G is a finite subgroup of the multiplicative group of a field then G is cyclic.*

Proof. We note that this result follows from the following observation:

Let G be a finite group with n elements. If for every $d|n$ we have $\left|\{x \in G | x^d = 1\}\right| \leq d$, then G is cyclic.

Indeed, if F is a field then the number of solutions of a polynomial of degree d is at most d.

We now prove the observation. Fix $d|n$ and consider the set G_d consisting of the elements of G with order d. Suppose that $G_d \neq \emptyset$ and let $y \in G_d$. Then

$$\langle y \rangle = \{y^j | j \in \mathbb{Z}\} \subset \{x \in G | x^d = 1\}.$$

But the subgroup $\langle y \rangle$ has cardinality d so $\langle y \rangle = \{x \in G | x^d = 1\}$. Therefore G_d is the set of generators of the cyclic group $\langle y \rangle$ of order d. We have shown that $|G_d| = \varphi(d)$ with $\varphi(d)$ Euler's totient function (the number of positive integers less than d relatively prime to d).

We have proved that G_d is empty or has cardinality $\varphi(d)$, for every $d|n$. So we have

$$n = |G| = \sum_{d|n} G_d \leq \sum_{d|n} \varphi(d) = n.$$

Therefore $|G_d| = \varphi(d)$ for every d dividing n . In particular $G_d \neq \emptyset$. This proves that G is cyclic. ■

The next result is due to Vinberg.

Lemma 50 *Let V be a finite dimensional vector space over $\mathbb{C}$ and assume that G is a finite subgroup of $GL(V)$ such that $(V - \{0\})/G$ is simply connected in the quotient topology. G is generated by its elements that have non-zero fixed points.*

Proof. Let H be the subgroup of G generated by those elements with non-zero fixed points. Then H is a normal subgroup of G. Hence

$$(V - \{0\})/G = ((V - \{0\})/H)/(G/H).$$

So $(V - \{0\})/H$ is a covering space of $(V - \{0\})/G$ with the group of deck transformations G/H. But $(V - \{0\})/G$ is simply connected so we must have $G/ = H$. ■

The standard result is

Proposition 51 *Let G be reductive and connected. Assume that $x \in Lie(G)$ is semi-simple. Then $C_G(x)$ is connected.*

Proof. Let T be the Zariski closure of $\{e^{tx} | t \in \mathbb{C}\}$ in G. Then T is an algebraic torus. The result now follows from Corollary 11.12 in [B] page 152. ■