that is

$$L\left(f_p g_p\right) \neq f(p) L g_p + g(p) L f_p.$$

Thus $T_p(X)$ makes sense for any variety X and $p \in X$.

Theorem. If $(X, \mathcal{O}_X)$ is an irreducible variety and $(Y, \mathcal{O}_Y)$ is a closed subvariety if $\dim Y = \dim X$ then $Y = X$.

$V \simeq \left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}\right)$

S_V reflection.

$M \simeq (1\,2\,3 \cdots 8)$. $\quad C = M S_V$

$\operatorname{order}(C) \simeq 30$ $\quad \eta \simeq e^{2\pi i/30}$

$1, 7, 11, 13, 17, 19, 23, 29$

C^{10} $\quad (9-1)(1+9^2)^4$

Lemma. Let X and Y be Zariski closed and disjoint subsets of $\mathbb{C}^n$ then there exists $\xi \in \mathcal{O}(\mathbb{C}^n)$ such that $\xi(X) = \{1\}$ and $\xi(Y) = \{0\}$.

Proof. Let

$$I_Y = \{\varphi \in \mathcal{O}(\mathbb{C}^n) | \varphi_{|Y} = 0\}.$$

Consider

$$J = \{\varphi_{|X} | \varphi \in I_Y\}.$$

Then if $x \in X$ then there exists $\psi \in J$ such that $\psi(x) \neq 0$. The nullstellensatz implies that $J = \mathcal{O}(X)$. Thus $1 \in J$. ■

Theorem. Let G be a reductive algebraic group and let (ρ, V) be a regular representation of G. If the orbits $\rho(G)v$ and $\rho(G)w$ are closed and not equal (i.e disjoint). Then there exists $f \in \mathcal{O}(V)^G$ such that

$$f(\rho(G)v) = \{1\}, f(\rho(G)w) = \{0\}.$$

Proof. Let K be a Z–dense compact subgroup of G. Let ξ be as in the Lemma above for

$$X = \rho(G)v, Y = \rho(G)w$$

and set

$$f(x) = \int_K \xi(\rho(k)x)dk.$$

Then

$$f(\rho(k)x) = f(x), k \in K, x \in V.$$

Hence

$$f(\rho(g)x) = f(x), g \in G, x \in V.$$

So it has the desired properties. ■

Let G be a reductive group, let (ρ, V) be a regular representation of G and let $\mathcal{O}(V)^G$ be the algebra of invariants. If $f_1, ..., f_m$ is a set of generators for $\mathcal{O}(V)^G$ then we have a morphism

$$F : V \to \mathbb{C}^m,$$

$$F(x) = (f_1(x), ..., f_m(x)).$$

Let X denote the Zariski closure of $F(V)$. Then we have a map of the set of G–orbits of V to X. We note that

$$F^*\mathcal{O}(X) = \mathcal{O}(V)^G.$$

Thus $(X, \mathcal{O}(X))$ is isomorphic with the variety $(spec_{\max}(\mathcal{O}(V)^G)$

We also note that

Lemma. $F(V) = X$.

We can thus think of X as the image of the set of orbits Gx.

Lemma. Let G be affine algebraic then the Zariski closure of Gx contains a closed orbit. If G is reductive then the orbit is unique.

Proof. We first prove the existance. Since G only has a finite number of connected components we may assume that G is connected. Let y be in the closure be such that $\dim Gy$ is minimal. We note that the closure of Gy is

irreducible. Suppose that Gy is not closed. Then there must be another orbit Gz in the closure of Gy. But then $\dim Gz < \dim Gy$.

Suppose that there are two closed orbits Gy and Gw in the closure of Gx.Then there exists $f \in \mathcal{O}(V)^G$ such that $f(Gy) = \{1\}$ and $f(Gw) = \{0\}$. But invariants are constant on orbit closures. ■

This implies that if G is reductive we can consider $spec_{\max}(\mathcal{O}(V)^G$ to be the set of closed G–orbits.

We have seen that if X is Z-closed and in $\mathbb{C}^n$ and $U \subset X$ is Z–open then the closure of U in the Z–topology is the same as that in the metric topology.

gXg^{-1}.

Lemma. If $X \in M_n(\mathbb{C})$ and $\mathrm{Ad}(GL(n,\mathbb{C}))X$ is closed if and only if X is semisimple.

Proof. If X is not semi-simple then $X = X_S + X_N$ and $X_N \neq 0$. There therefore exists $h \in M_n(\mathbb{C})$ such that $[h, X_N] = 2X_N, [h, X_S[= 0$. Thus setting $\phi(t) = e^{-th}$. Then $\phi(t)X\phi(t)^{-1} = X_S + e^{-2t}X_N$. Thus X_S is in the closure of $Ad(GL(n,\mathbb{C}))X$. So the orbit is not closed. Thus closed orbit implies semi-simple.

We note that if X and Y have the same characteristic polynomial and if X is semi-simple and Y is not then

$$\dim C_{GL(n,C)}(Y) \leq \dim C_{GL(n,C)}(X).$$

Indeed, $Y = Y_S + Y_N$ if $gYg^{-1} = Y$ then $gY_Sg^{-1} = Y_S$ and $gY_Ng^{-1} = Y_n$. This implies that

$$\dim C_{GL(n,C)}(Y) = C_{C_{GL(n,C)}(Y_S)}(Y_N) \leq \dim C_{GL(n,C)}(Y_S).$$

If X is semi-simple and has distinct eigenvalues $\lambda_1, ..., \lambda_r$ and if m_i is the multiplicity of λ_i then

$$\dim C_{GL(n,C)}(X) = \sum m_i^2.$$

Assume that $X \in M_n(\mathbb{C})$ is semi-simple let Z be the closure of $Ad(GL(n,\mathbb{C}))X$. We will see that the orbits of minimal dimension are closed in Z. This implies that X is closed. ■

Lemma. (Richardson) Let $G \subset GL(n,\mathbb{C})$ be Z–closed and assume that H is a Zariski closed connected subgroup of G. Assume that $V \subset \mathbb{C}^n$ is a H–invariant subspace such that if $v \in V$ then $\mathsf{Lie}(G)v \cap V = \mathsf{Lie}(H)v$. If $x \in V$ then $Gx \cap V$ is a disjoint union of a finite number of orbits of H all of which are of the same dimension and are open in $Gx \cap V$.

Proof. Let $u \in Gx \cap V$. Then

$$T_u(Gx \cap V) \subset \mathsf{Lie}(G)u \cap V = \mathsf{Lie}(H)u.$$

Also, since

$$Hu \subset Gu \cap V = Gx \cap V$$

$T_u(G \cdot u) = \mathrm{Lie}(G)u \quad \frac{d}{dt}e^{tX}u|_{t=0} = Xu$

we see that

$$\dim(Gx \cap V) = \dim(Hu)$$

This implies that the orbits of H in $Gx \cap V$ all have the same dimension and are open in $Gx \cap V$. Hence they are closed and since they are irreducible they are the irreducible components of the variety $Gx \cap V$. ■

A simple consequence of this lemma are the following well known finiteness result and the criterion for an adjoint orbit to be closed.

Theorem. Let $\mathfrak{g} \subset M_n(\mathbb{C})$ be the Lie algebra of a reductive Z–closed subgroup, G, of $GL(n, \mathbb{C})$. Let $\mathcal{N}_{\mathfrak{g}}$ denote the subvariety of nilpotent elements in $\mathfrak{g}$. Then the number of $\mathsf{Ad}(G)$ orbits in $\mathcal{N}_{\mathfrak{g}}$ is finite.

Proof. We note that the nilpotent elements in $M_n(\mathbb{C})$ are parametrized up to conjugacy by their Jordan decompositions and there is one for each partition of n. Let

$$M_n(\mathbb{C}) = \mathfrak{g} \oplus Z$$

with $\mathsf{Ad}(G)Z = Z$ (here Ad is the action $\mathrm{Ad}(g)X = gXg^{-1}$ of $GL(n,\mathbb{C})$ on $M_n(\mathbb{C})$). Then if $X \in \mathfrak{g}$ and

$$Y \in [M_n(\mathbb{C}), X] \cap \mathfrak{g}$$

then

$$Y = [U + W, X]$$

with $U \in \mathfrak{g}$ and $W \in Z$. Thus $[U, X] \in \mathfrak{g}$ and $[W, X] \in Z$. Hence $Y = [U, X]$. So the hypothesis of the above lemma is satisfied. This implies that if $X \in \mathcal{N}_{\mathfrak{g}}$ then $\mathsf{Ad}(GL(n,\mathbb{C}))X \cap \mathfrak{g}$ is a finite union of open $\mathsf{Ad}(G)$–orbits. ■

Lemma. Let $\mathfrak{g}$ and G be as in the previous theorem. If $X \in \mathfrak{g}$ then $\mathsf{Ad}(G)X$ is closed if and only if X is semi-simple.

Proof. As in the proof of the previous Theorem. The conditions of Richardson's lemma are satisfied for $\mathsf{Ad}(G)$ and $\mathsf{Ad}(GL(n,\mathbb{C}))$ both thought of as subgroups of $GL(M_n(\mathbb{C}))$. If $X \in \mathfrak{g}$ is semisimple then $\mathsf{Ad}(GL(n,\mathbb{C}))X$ is closed.

Richardson's lemma implies that $\mathsf{Ad}(GL(n,\mathbb{C}))X \cap \mathfrak{g}$ is a finite disjoint union of relatively open $\mathsf{Ad}(G)$–orbits. Hence each of the $\mathsf{Ad}(G)$–orbits is closed.

If GX is closed we wish to show that X is semisimple. If not then $X = X_S + X_N$ and there exists $h \in M_n(\mathbb{C})$ such that $[h, X_S] = 0$ and $[h, X_N] = 2X_N$. Let $M_n(\mathbb{C}) = Lie(G) \oplus Z$ with $Ad(G)Z \subset Z$ and h_1 be the projection of h into $Lie(G)$. Thus $h = h_1 + z, z \in Z$. Now $2X_N = [h, X_N] = [h_1, X_N] + [z, X_N]$. But $[z, Lie(G)] \subset Z$. Thus $[z, X_N] = 0$. But then $X_S = \lim_{t->+\infty} e^{-tad(h_1)}X$ and $X_S \notin Ad(G)X$. ■