Let $G$ be an affine algebraic group and let $S$ be the set of $\mathbb{Z}$-closed connected solvable subgroups ordered by inclusion.

**Theorem.** There exists a maximal element in $S$ and all of the maximal elements are conjugate. Furthermore, $G/S$ is projective for each such $S$.

**Proof.** Let $S$ be a $\mathbb{Z}$–closed connected solvable subgroup of maximal dimension. If $S \subset S_1$ a connected solvable subgroup then since $\dim S_1 = \dim S$ we must have $S = S_1$.

We have shown that there exists an injective regular representation $(\rho, V)$ of $G$ and an element $v \in V - \{0\}$ such that $S$ is the stabilizer in $G$ of the line $[v]$. Let $X$ be the set of all flags in $V$

$$U : U_1 \subset U_2 \subset ...$$

such that $U_1 = [v]$. We assert that $X$ is $\mathbb{Z}$-closed. Let

$$\Phi : \mathcal{F}(V) \to \mathbb{P}(V)$$
be given by
\[ U_1 \subset U_2 \subset ... \longrightarrow U_1 \]
then \( \Phi \) is a morphism and \( X = \Phi^{-1}[v] \).

If \( V \in X \) and if \( g \in G \) is such that \( gV = V \) then \( g[v] = [v] \) hence \( g \in S \). Clearly \( S \) acts on \( X \) and hence it has a fixed point in \( X, \mathcal{V}_o \). Now \( G\mathcal{V}_o \) gives a realization of \( G/S \) as a \( Z \)-open subset of the Zariski closure of \( G\mathcal{V}_o \), \( Y \). We assert that \( G\mathcal{V}_o = Y \). Let \( GU \) be a closed orbit in \( Y \). Then the identity component of \( GU \) must have dimension at most \( \dim S \). Thus
\[ \dim GU \geq \dim G\mathcal{V}_o \]
hence
\[ GU = G\mathcal{V}_o. \]

Let \( S_1 \) be another maximal element in \( S \). Then \( S_1 \) must have a fixed point in \( Y, \mathcal{V}_o \). This implies that \( g^{-1}S_1g \) fixes \( \mathcal{V}_o \) and hence is contained in \( S \). ■
A maximal element of $S$ is called a Borel subgroup.

**Theorem.** If $S$ is a solvable, affine algebraic group then the subset of unipotent elements, $U$, forms a $Z$-closed, normal subgroup.

**Proof.** We may assume that $S \subset GL(n, \mathbb{C})$ as a $Z$-closed subgroup. If $g$ is unipotent then $g \in S^o$ the identity component of $S$. Since $S^o$ has a fixed point in $F_n$ we may assume that $S^o$ is contained in $B_n$ the group of upper triangular elements of $GL(n, \mathbb{C})$. Let $U_n$ be the subgroup of $B_n$ consisting of the elements with ones on the main diagonal. Then

$$U = S^o \cap U_n$$

which is a $Z$–closed subgroup. Since any conjugate of a unipotent element is unipotent the result follows. ■

If $G$ is an affine algebraic group then we define the unipotent radical to be the union of all $Z$–closed unipotent normal subgroups of $G$. 
**Exercise.** The unipotent radical is a $Z$–closed, unipotent, normal subgroup.

**Lemma.** Let $G$ be an affine algebraic group with unipotent radical $U$. Then $U$ acts trivially on any irreducible, regular, representation of $G$.

**Proof.** Let $(\rho, V)$ be an irreducible, regular, non-zero, representation. Since $U$ is solvable and connected, there is a basis of $V$ such that $\rho(U)$ consists of upper triangular matrices with ones on the main diagonal. This implies that $V^U \neq 0$. Since $U$ is normal $V^U$ is $G$–invariant. ■

**Theorem.** If $G$ is $Z$–closed subgroup of $GL(n, \mathbb{C})$ acting completely reducibly on $\mathbb{C}^n$ then $G$ is linearly reductive.

**Theorem.** Let $G$ be an affine algebraic group then $G$ is linearly reductive if and only if its unipotent radical is trivial.
Proof. Suppose that $G$ is not reductive. We may assume that $G \subset GL(n, \mathbb{C})$ as a $\mathbb{Z}$–closed subgroup. Let

$$\mathbb{C}^n = V_1 \supsetneq V_2 \supsetneq \ldots \supsetneq V_m \supsetneq V_{m+1} = \{0\}$$

be a composition series for the representation. Set $Z_i = V_i / V_{i+1}$ then we have a representation, $\mu$, of $G$ on

$$Z = Z_1 \oplus Z_2 \oplus \ldots \oplus Z_m.$$ 

If $\ker \mu = \{e\}$ then $\mu(G)$ is linearly reductive and isomorphic with $G$. If $g \in \ker \mu$ then

$$(\mu(g) - I)V_i \subset V_{i+1}$$

for all $i$. This implies $\ker \mu$ is a normal subgroup that consists of unipotent elements. Hence trivial unipotent radical implies linearly reductive.

If $G \subset GL(n, \mathbb{C})$ is linearly reductive. Then

$$\mathbb{C}^n = V_1 \oplus V_2 \oplus \ldots \oplus V_m$$

with $V_i$ irreducible. The unipotent radical of $G$ acts trivially on each of the $V_i$ and hence on $\mathbb{C}^n$. ■
Theorem. Let $H$ be a $Z$–closed subgroup of an affine algebraic group $G$ over $\mathbb{C}$. Then the following are equivalent:

Theorem 1 1. $H$ contains a Borel subgroup.

2. $G/H$ is compact in the $S$–topology.

3. $G/H$ is projective.

Under any of these conditions we will call $H$ a parabolic subgroup of $G$.

Proof. We have seen that a quasi-projective variety is compact in the $S$–topology if and only if it is projective. Thus 2 and 3 are equivalent. If $G/H$ is projective then the Borel fixed point theorem implies that if $B$ is a Borel subgroup of $G$ then $B$ has a fixed point in $G/H$. This implies that $B$ is conjugate to a subgroup of $H$. Since a conjugate of a Borel subgroup is a Borel subgroup 3.
implies 1. If $H$ contains a Borel subgroup then we have a surjective regular map $f : G/B \to G/H$. Since $G/B$ is projective it is compact and $f$ is continuous in the $S$–topology we see that if $B \subset H$ then $G/H$ is compact in the $S$–topology. Thus 1. implies 2. □

**Exercise.** Is there a similar theorem if we only assume that $H$ is $S$–closed in $G$?

Let $G$ be a reductive subgroup of $GL(n, \mathbb{C})$ and let $H$ be a Cartan subgroup of $G$. Set $\mathfrak{h} = Lie(H)$ and let $\Phi$ be the root system of $\mathfrak{g} = Lie(G)$ with respect to $H$. Let $\mathfrak{h}_\mathbb{R}$ be the span of the coroots. If $h \in \mathfrak{h}_\mathbb{R}$ then $\alpha(h) \in \mathbb{R}$ for all $\alpha \in \Phi$. There exists $h \in \mathfrak{h}_\mathbb{R}$ such that $\alpha(h) \neq 0$ if $\alpha \in \Phi$.

**Proposition.** Let

$$\mathfrak{b} = \mathfrak{h} \bigoplus \left( \bigoplus_{\alpha \in \Phi, \alpha(h) > 0} \mathfrak{g}_\alpha \right).$$
Then there exists a Borel subgroup $B$ in $G$ such that $b$ is $\text{Lie}(B)$.

**Proof.** We note that $b_1 = [b, b] = \bigoplus_{\alpha \in \Phi, \alpha(h)>0} g_\alpha$. Let $a_1 = \min_{\alpha \in \Phi, \alpha(h)>0} \alpha(h)$ then

$$b_2 = [b_1, b_1] \subset \bigoplus_{\alpha \in \Phi, \alpha(h)>a_1} g_\alpha.$$  

If we define $a_2 = \min_{\alpha \in \Phi, \alpha(h)>a_1} \alpha(h)$ then

$$b_3 = [b_2, b_2] \subset \bigoplus_{\alpha \in \Phi, \alpha(h)>a_2} g_\alpha.$$  

Thus procedure will eventually lead to 0. Thus if $H$ is the connected Lie subgroup of $G$ then $H$ is solvable. The $Z$–closure, $H_1$, of $H$ is solvable and the identity component of $H_1$ contains $B$. We assert that

$$\text{Lie}(H_1) = \text{Lie}(H).$$

Indeed if $\text{Lie}(H_1) \supsetneq b$ then there must be

$$g_\alpha \subset \text{Lie}(H_1), \alpha(h) < 0.$$  

Hence $\text{Lie}(H_1)$ contains a TDS. This is a contradiction. Hence $H_1 = B$. The same argument shows that $B$ is maximal connected solvable.  ■