Let G be an affine algebraic group and let $\mathcal{S}$ be the set of Z-closed connected solvable subgroups ordered by inclusion.

Theorem. There exists a maximal element in $\mathcal{S}$ and all of the maximal elements are conjugate. Furthermore, G/S is projective for each such S.

Proof. Let S be a Z–closed connected solvable subgroup of maximal dimension. If $S \subset S_1$ a connected solvable subgroup then since $\dim S_1 = \dim S$ we must have $S = S_1$.

We have shown that there exists an injective regular representation (ρ, V) of G and an element $v \in V - \{0\}$ such that S is the stabilizer in G of the line $[v]$. Let X be the set of all flags in V

$$\mathcal{U} : U_1 \subset U_2 \subset \ldots$$

such that $U_1 = [v]$. We assert that X is Z-closed. Let

$$\Phi : \mathcal{F}(\mathcal{V}) \to \mathbb{P}(V)$$

be given by

$$U_1 \subset U_2 \subset \ldots \longmapsto U_1$$

then Φ is a morphism and $X = \Phi^{-1}[v]$.

If $\mathcal{V} \in X$ and if $g \in G$ is such that $g\mathcal{V} = \mathcal{V}$ then $g[v] = [v]$ hence $g \in S$. Clearly S acts on X and hence it has a fixed point in X, $\mathcal{V}_o$. Now $G\mathcal{V}_o$ gives a realization of G/S as a Z–open subset of the Zarisky closure of $G\mathcal{V}_o$, Y. We assert that $G\mathcal{V}_o = Y$. Let $G\mathcal{U}$ be a closed orbit in Y. Then the identity component of $G_{\mathcal{U}}$ must have dimension at most $\dim S$. Thus

$$\dim G\mathcal{U} \geq \dim G\mathcal{V}_o$$

hence

$$G\mathcal{U} = G\mathcal{V}_o.$$

Let S_1 be another maximal element in $\mathcal{S}$. Then S_1 must have a fixed point in Y, $g\mathcal{V}_o$. This implies that $g^{-1}S_1g$ fixes $\mathcal{V}_o$ and hence is contained in S. $\blacksquare$

A maximal element of $\mathcal{S}$ is called a Borel subgroup.

Theorem. If S is a solvable, affine algebraic group then the subset of unipotent elements, U, forms a Z-closed, normal subgroup.

Proof. We may assume that $S \subset GL(n, \mathbb{C})$ as a Z–closed subgroup. If g is unipotent then $g \in S^o$ the identity component of S. Since S^o has a fixed point in $\mathcal{F}_n$ we may assume that S^o is contained in B_n the group of upper triangular elements of $GL(n, \mathbb{C})$. Let U_n be the subgroup of B_n consisting of the elements with ones on the main diagonal. Then

$$U = S^o \cap U_n$$

which is a Z–closed subgroup. Since any conjugate of a unipotent element is unipotent the result follows. ■

If G is an affine algebraic group then we define the unipotent radical to be the union of all Z–closed unipotent normal subgroups of G.

Exercise. The unipotent radical is a Z–closed, unipotent, normal subgroup.

Lemma. Let G be an affine algebraic group with unipotent radical U. Then U acts trivially on any irreducible, regular, representation of G.

Proof. Let (ρ, V) be an irreducible, regular, non-zero, representation. Since U is solvable and connected, there is a basis of V such that $\rho(U)$ consists of upper triangular matrices with ones on the main diagonal. This implies that $V^U \neq 0$. Since U is normal V^U is G–invariant. ∎

Theorem. If G is Z–closed subgroup of $GL(n, \mathbb{C})$ acting completely reducibly on $\mathbb{C}^n$ then G is linearly reductive.

Theorem. Let G be an affine algebraic group then G is linearly reductive if and only if its unipotent radical is trivial.

Proof. Suppose that G is not reductive. We may assume that $G \subset GL(n, \mathbb{C})$ as a Z–closed subgroup.Let

$$\mathbb{C}^n = V_1 \supsetneqq V_2 \supsetneqq \dots \supsetneqq V_m \supsetneqq V_{m+1} = \{0\}$$

be a composition series for the repesentation. Set $Z_i = V_i/V_{i+1}$ then we have a representation, μ,of G on

$$Z = Z_1 \oplus Z_2 \oplus \dots \oplus Z_m.$$

If $\ker \mu = \{e\}$ then $\mu(G)$ is linearly reductive and isomorphic with G. If $g \in \ker \mu$ then

$$(\mu(g) - I)V_i \subset V_{i+1}$$

for all i. This implies $\ker \mu$ is a normal subgroup that consists of unipotent elements. Hence trivial unipotent radical implies linearly reductive.

If $G \subset GL(n, \mathbb{C})$ is linearly reductive. Then

$$\mathbb{C}^n = V_1 \oplus V_2 \oplus \dots \oplus V_m$$

with V_i irreducible. The unipotent radical of G acts trivially on each of the V_i and hence on $\mathbb{C}^n$. ■

Theorem. Let H be a Z–closed subgroup of an affine algebraic group G over $\mathbb{C}$. Then the following are equivalent:

Theorem 1 *1. H contains a Borel subgroup.*

2. G/H is compact in the S–topology.

3. G/H is projective.

Under any of these conditions we will call H a parabolic subgroup of G.

Proof. We have seen that a quasi-projective variety is compact in the S–topology if and only if it is projective. Thus 2 and 3 are equivalent. If G/H is projective then the Borel fixed point theorem implies that if B is a Borel subgroup of G then B has a fixed point in G/H. This implies that B is conjugate to a subgroup of H. Since a conjugate of a Borel subgroup is a Borel subgroup 3.

implies 1. If H contains a Borel subgroup then we have a surjective regular map $f : G/B \to G/H$. Since G/B is projective it is compact and f is continuous in the S–topology we see that if $B \subset H$ then G/H is compact in the S–topology. Thus 1. implies 2. ■

Exercise. Is there a similar theorem if we only assume that H is S–closed in G?

Let G be a reductive subgroup of $GL(n, \mathbb{C})$ and let H be a Cartan subgroup of G. Set $\mathfrak{h} = Lie(H)$ and let Φ be the root system of $\mathfrak{g} = Lie(G)$ with respect to H. Let $\mathfrak{h}_{\mathbb{R}}$ be the span of the coroots. If $h \in \mathfrak{h}_{\mathbb{R}}$ then $\alpha(h) \in \mathbb{R}$ for all $\alpha \in \Phi$. There exists $h \in \mathfrak{h}_{\mathbb{R}}$ such that $\alpha(h) \neq 0$ if $\alpha \in \Phi$.

Proposition. Let

$$\mathfrak{b} = \mathfrak{h} \bigoplus \left(\bigoplus_{\alpha \in \Phi, \alpha(h) > 0} \mathfrak{g}_\alpha \right).$$

Then there exists a Borel subgroup B in G such that $\mathfrak{b}$ is $Lie(B)$.

Proof. We note that $\mathfrak{b}_1 = [\mathfrak{b}, \mathfrak{b}] = \bigoplus_{\alpha\in\Phi,\alpha(h)>0} \mathfrak{g}_\alpha$. Let $a_1 = \min_{\alpha\in\Phi,\alpha(h)>0} \alpha(h)$ then

$$\mathfrak{b}_2 = [\mathfrak{b}_1, \mathfrak{b}_1] \subset \bigoplus_{\alpha\in\Phi,\alpha(h)>a_1} \mathfrak{g}_\alpha.$$

If we define $a_2 = \min_{\alpha\in\Phi,\alpha(h)>a_1} \alpha(h)$ then

$$\mathfrak{b}_3 = [\mathfrak{b}_2, \mathfrak{b}_2] \subset \bigoplus_{\alpha\in\Phi,\alpha(h)>a_2} \mathfrak{g}_\alpha.$$

Thus procesure will eventually lead to 0. Thus if H is the connected Lie subgroup of G then H is solvable. The Z–closure, H_1, of H is solvable and the identity component of H_1 contains B. We assert that

$$Lie(H_1) = Lie(H).$$

Indeed if $Lie(H_1) \underset{\neq}{\supset} \mathfrak{b}$ then there must be

$$\mathfrak{g}_\alpha \subset Lie(H_1), \alpha(h) < 0.$$

Hence $Lie(H_1)$ contains a TDS. This is a contradiction. Hence $H_1 = B$. The same argument shows that B is maximal connected solvable. ■