At this point we have proved \((W \subset GL(n, \mathbb{C})\) then \(R(W)\) is the set of complex reflections in \(W\)).

**Lemma** If \(W\) is a finite subgroup \(GL(n, \mathbb{C})\) such that \(O(\mathbb{C}^n)^W\) has \(n\), algebraically independent, homogeneous, generators of degrees \(d_1, \ldots, d_n\) then

1. \(d_1 \cdots d_n = |W|\)

2. \(|R(W)| = \sum_{j=1}^{n} (d_j - 1)\).

We are now ready to prove the enhancement of 1. \(\implies\) 2.

Assume that we have \(g_1, \ldots, g_n\) algebraically independent homogeneous elements of \(O(V)^W\) with \(\deg g_i = e_i\).

Molien’s formula implies that

\[
\frac{1}{|W|} \sum_{s \in W} \frac{1}{\det(1 - qs)} = \sum_{j \geq 0} q^j \dim O^j(\mathbb{C}^n)^W.
\]
Also

\[ \dim \mathbb{C}^d[g_1, \ldots, g_n] = \left| \{ I \mid \sum i_j e_j = d \} \right| = a_j. \]

Clearly \( a_j \leq \dim \mathcal{O}^j(\mathbb{C}^n)^W \). Thus for \( 0 < q < 1 \) we have

\[ \frac{1}{|W|} \sum_{s \in W} \frac{(1 - q)^n}{\det(1 - qs)} \geq \prod_{j=1}^n \frac{1 - q}{1 - q^{e_j}} \]

taking the limit as \( q \to 1 \) we have

\[ \frac{1}{|W|} \geq \frac{1}{e_1 \cdots e_n}. \]
Now suppose that $|W| = e_1 \cdots e_n$ and $|q| < 1$ then

$$\frac{1}{|W|} \sum_{s \in W \setminus \{I\}} \frac{(1 - q)^n}{\det(1 - qs)} =$$

$$\frac{1}{|W|} \sum_{s \in W} \frac{(1 - q)^n}{\det(1 - qs)} - \frac{1}{|W|} \geq \prod_{j=1}^{n} \frac{1 - q}{1 - q^{e_j}} - \frac{1}{|W|}.$$ 

Hence

$$\frac{|R(W)|}{2} = \lim_{\substack{q \to 1 \\ q < 1}} \left( \sum_{s \in W} \frac{(1 - q)^n}{\det(1 - qs)} - 1 \right) \geq$$

$$\lim_{\substack{q \to 1 \\ q < 1}} \frac{\left( |W| \prod_{j=1}^{n} \frac{1 - q^{e_j}}{1 - q} - 1 \right)}{1 - q} = \frac{1}{2} \sum_{i=1}^{n} (e_i - 1).$$
If $W'$ is the subgroup generated by $R(W)$ then $\mathcal{O}(\mathbb{C}^n)^{W'}$ is generated by $f_1, \ldots, f_n$, homogeneous, algebraically independent elements of $\mathcal{O}(\mathbb{C}^n)^{W'}$. Since $\mathcal{O}(\mathbb{C}^n)^W \subset \mathcal{O}(\mathbb{C}^n)^{W'}$ if $d_i = \deg f_i$ and $d_1 \leq \ldots \leq d_n$ then
\[d_i \leq e_i, i = 1, \ldots, n.\]

However,
\[
\frac{|R(W)|}{2} = \frac{|R(W')|}{2} = \frac{1}{2} \sum (d_i - 1).
\]
This implies that
\[
\frac{1}{2} \sum (d_i - 1) \geq \frac{1}{2} \sum_{i=1}^{n} (e_i - 1).
\]
Hence $e_i = d_i$. Hence
\[
|W'| = d_1 \cdots d_n = e_1 \cdots e_n = |W|.
\]
So $W = W'$. Also since $e_i = d_i$ we see that $\mathbb{C}[g_1, \ldots, g_n] = \mathbb{C}[f_1, \ldots, f_n] = \mathcal{O}(\mathbb{C}^n)^W$. 
Examples. 1. Taking $g_j = \sum_{i=1}^{n} x_i^j$. Then $g_1, \ldots, g_n$ are algebraically independent and $\deg g_j = j$. So

$$O(C^n)^{S_n} = \mathbb{C}[g_1, \ldots, g_n].$$

2. $W$ the group of signed permutations acting on $\mathbb{C}^n$. Then $|W| = 2^n n!$. $W$ is generated by the transpositions and the reflections, $\varepsilon_i$, about the hyperplanes $x_i = 0$. That is, $\varepsilon_j$ is diagonal with diagonal entries 1 except in the $i$–th position and a $-1$ in the $i$–position.

Set $g_j = \sum_{i=1}^{n} x_i^{2j}$. Then $g_1, \ldots, g_n$ are algebraically independent and $\deg g_j = 2j$. Thus $\mathbb{C}[g_1, \ldots, g_n] = O(C^n)^W$.

3. $W$ the group of signed permutations acting on $\mathbb{C}^n$ with an even number of sign changes. Let $\varepsilon_i$ be the Then $W$ is generated by the transpositions and the reflections $\varepsilon_i \varepsilon_j(i,j)$ for $i \neq j$. Let $g_j = \sum_{i=1}^{n} x_i^{2j}$, $i = 1, \ldots, n - 1$ and $g_n = x_1 \cdots x_n$. $e_1 \cdots e_n = 2^{n-1} n!$ hence $\mathbb{C}[g_1, \ldots, g_n] = O(C^n)^W$.

This describes generating sets of invariants for the classical Weyl groups. $W_{A_n}, W_{B_n}(= W_{C_n}), W_{D_n}$. 
Chevalley’s argument to prove 1. in the theorem.

$W$ a finite subgroup of $GL(n, \mathbb{C})$

$u_1, \ldots, u_r$ be a minimal set of generators for the ideal $O(\mathbb{C}^n)O(\mathbb{C}^n)W$.

$d_i = \deg u_i$ and $t_1, \ldots, t_r$ be indeterminates with assigned degrees $d_1, \ldots, d_r$.

$\varphi$ a homogeneous polynomial of minimal positive weighted degree such that $\varphi(u_1, \ldots, u_r) = 0$.

Let $\varphi_i = \frac{\partial \varphi}{\partial t_i}$. If $\varphi_i(u_1, \ldots, u_r) = 0$ then $\varphi_i = 0$.

Let $\varphi_i \neq 0$ for $i \leq m$ and $\varphi_i = 0$ for $i > m$.

Set $q_i = \varphi_i(u_1, \ldots, u_r)$ for $i \leq m$. Then

$$0 = \frac{\partial \varphi(u_1, \ldots, u_r)}{\partial x_j} = \sum_{i=1}^{m} q_i \frac{\partial u_i}{\partial x_j}.$$
After reordering we assume that $q_1, \ldots, q_l$ form a minimal subset of \{q_1, \ldots, q_m\} generating $\sum_{i=1}^{m} \mathcal{O}(\mathbb{C}^n) q_i$.

That is if $l < i \leq m$ then we have

$$q_i = \sum_{k=1}^{l} h_{ik} q_k$$

with $h_{ij} \in \mathcal{O}(\mathbb{C}^n)$ and homogenous.
Substituting
\[ \sum_{i=1}^{l} q_i \frac{\partial u_i}{\partial x_j} + \sum_{i=l+1}^{m} \sum_{k=1}^{l} h_{ik} q_k \frac{\partial u_i}{\partial x_j} = 0. \]

This yields
\[ \sum_{i=1}^{l} q_i \left( \frac{\partial u_i}{\partial x_j} + \sum_{p=l+1}^{m} h_{pi} \frac{\partial u_p}{\partial x_j} \right) = 0 \]

If we set
\[ v_{ij} = \frac{\partial u_i}{\partial x_j} + \sum_{p=l+1}^{m} h_{pi} \frac{\partial u_p}{\partial x_j} \]
then \( v_{ij} \) is homogeneous of degree \( d_i - 1 \) and so
\[ q_i v_{ij} \in \sum_{k \neq i, k \leq l} O(\mathbb{C}^n) q_k. \]

Suppose we could prove that this implies that \( v_{ij} \in O(\mathbb{C}^n)O(\mathbb{C}^n)_W^+. \) Then
\[ \frac{\partial u_i}{\partial x_j} + \sum_{p=l+1}^{m} h_{pi} \frac{\partial u_p}{\partial x_j} = \sum w_{ijk} u_k. \]
Noting that $i \leq l$ and we can discard all homogeneous terms degree $\neq d_i - 1$ we see that after throwing away those terms the right hand side is

$$\sum_{k \neq i} w_{ijk} u_k.$$ 

Multiply both sides of this equation by $x_j$ and sum in $j$ we have

$$\sum_j x_j \frac{\partial u_i}{\partial x_j} + \sum_{p = l+1}^m h_{pi} \sum_j x_j \frac{\partial u_p}{\partial x_j} = \sum_{k \neq i} \left( \sum_j x_j w_{ijk} \right) u_k.$$ 

If $\varphi$ is homogeneous of degree $k$ then

$$\sum_j x_j \frac{\partial \varphi}{\partial x_j} = k \varphi.$$ 

We therefore have

$$d_i u_i = - \sum_{p = l+1}^m h_{pi} d_p u_p + \sum_{k \neq i} \left( \sum_j x_j w_{ijk} \right) u_k.$$ 

This contradicts the minimality of $u_1, \ldots, u_r$.

Here is Chevalley’s lemma which completes the proof.
Lemma. Let $W \subset GL(n, \mathbb{C})$ be a finite complex reflection group and let $u_1, \ldots, u_r$ be homogenous elements of $O(\mathbb{C}^n)^W$. Suppose that $u \in O(\mathbb{C}^n)^W$ and $v \in O(\mathbb{C}^n)$ are homogenous and that $u \notin \sum O(\mathbb{C}^n)u_i$ and that $uv \in \sum O(\mathbb{C}^n)u_i$ then $v \in O(\mathbb{C}^n)O(\mathbb{C}^n)^W$.

$s$ a complex reflection then $\dim \ker(I - s) = n - 1$ so $\ker(I - s) = \ker \lambda$ with $\lambda \in (\mathbb{C}^n)^*$. Let $u \in \mathbb{C}^n$ satisfy $\lambda(u) = 1$ and $su = \zeta u$. If $x \in \mathbb{C}^n$ then $x - \lambda(x)u \in \ker \lambda$ so

$sx = s(x - \lambda(x)u) + \lambda(x)su = x - \lambda(x)u + \zeta \lambda(x)u$

That is

$sx = x + (\zeta - 1)\lambda(x)u$. 
Exercise. If $\lambda$ is in $(\mathbb{C}^n)^*$, if $u \in \mathbb{C}^n$ is such that $\lambda(u) = 1$ and $\zeta$ is a primitive $m$–th, root of 1 with $m \neq 1$. Defining

$$s_{\lambda,\zeta,u}(x) = x + (\zeta - 1)\lambda(x)u$$

then $s_{\lambda,\zeta}$ is a complex reflection.
Proof of Lemma by induction on degree \( v \). If \( \deg v = 0 \) then \( v \) must be zero by the hypothesis.

If \( s \) is a complex reflection, such that \( s = s_{\lambda, \zeta, u} \) then if \( w \in \mathcal{O}(\mathbb{C}^n) \) then \((I - s^*)w = \lambda \tilde{w}\). If \( h \in \mathcal{O}(\mathbb{C}^n)_+^W \) then \((1 - s^*)hw = hw - s^*hs^*w = h(I - s^*)w\). So
\[
u(I - s^*)v \in \sum ((1 - s^*)\mathcal{O}(\mathbb{C}^n)) u_i
\]
So both sides of the equation are divisible by \( \lambda \) thus
\[
u \tilde{v} \in \sum \mathcal{O}(\mathbb{C}^n) u_i.
\]
So the inductive hypothesis implies
\[
\tilde{v} \in \mathcal{O}(\mathbb{C}^n)\mathcal{O}(\mathbb{C}^n)_+^W.
\]
Hence
\[
(I - s^*)v \in \mathcal{O}(\mathbb{C}^n)\mathcal{O}(\mathbb{C}^n)_+^W
\]
for all complex reflections in \( W \). This implies the same holds true for all \( s \in W \). We therefore have after summing both sides over \( s \in W \)
\[
|W| v \in \sum_{s \in W} s^* v + \mathcal{O}(\mathbb{C}^n)\mathcal{O}(\mathbb{C}^n)_+^W.
\]
**Note** This gives another necessary and sufficient condition that for a finite subgroup of $GL(n, \mathbb{C})$ to be a finite reflection group.