Lemma. If $W \subset GL(n, \mathbb{C})$ is a finite subgroup generated by complex reflections and if $f_1, ..., f_n$ is a set of homogeneous generators of $\mathcal{O}(\mathbb{C}^n)^W$ then $f_1, ..., f_n$ is a system of parameters of $\mathcal{O}(\mathbb{C}^n)$.

Proof. Suppose that $f_i(x) = 0$ for all $i = 1, ..., n$. Then $f(x) = 0$ for all $f \in \mathcal{O}(\mathbb{C}^n)^W$. Thus $x \in W0 = 0$. ■

This gives the more sophisticated proof of the freeness that was proved using Chevalley's Lemma.

If W is a finite subgroup of $GL(V)$ and ζ is a primitive m–th root of 1 and if $s \in W$ then set

$$U(s, \zeta) = \{v \in V | sv = \zeta v\}.$$

An element $s \in W$ is said to be regular if there exists primitive m–th root of 1, $\zeta \neq 1$ such that $U(s, \zeta)$ contains a regular element.

If $U(s\,\zeta)$ contains a regular element and ζ is a primitive m–th root of 1 then $s^m = I$.

We will now embark on the proof of the following theorem of Springer.

Theorem Let $W \subset GL(n, \mathbb{C})$ be a finite reflection group and let θ be a regular element of W of order m. Let $g_1, ..., g_n$ be an algebraically independent set of homogeneous generators of $\mathcal{O}(\mathbb{C}^n)^W$ arranged so that

$$m | \deg g_i, i = 1, ..., r, m \nmid \deg g_i, i > r.$$

Let ξ be a primitive m–th root of 1 and $\mathfrak{a} = U(\theta, \xi)$ then $W^{\theta}_{|\mathfrak{a}}$ is a complex reflection group and if $f_i = g_{i|\mathfrak{a}}$ then

$$\mathcal{O}(\mathfrak{a})^{W_\theta} = \mathbb{C}[f_1, ..., f_r].$$

θ a regular element of order m and ξ a primitive m–th root of unity such that $U(\theta, \xi)$ contains a regular element.

Lemma (1) If $s \in W$ and $U(\theta, \xi) \subset U(s, \xi)$ then $s = \theta$.

(2) If $s \in W$ and $sU(\theta, \xi) \subset U(\theta, \xi)$ then $s \in W^\theta$.

Proof. To prove (1) we note that $s^{-1}\theta_{|U(\theta,\xi)} = I$ thus $s = \theta$. For (2) note that $sU(\theta, \xi) = U(s\theta s^{-1}, \xi)$. Thus (1) implies (2). ■

Let $g_1, ..., g_n$ be homogeneous generators of $\mathcal{O}(\mathbb{C}^n)^W$ we assume that $m|\deg g_i, i = 1, ..., r$ and $m \nmid \deg g_i$, $i > r$.

Lemma $X_\theta = \cap_{i>r}\{v \in V | g_i(v) = 0\} = \cup_{s\in W} U(s, \xi)$.

Proof. Let $v \in \cap_{i>r}\{v \in V | g_i(v) = 0\}$ then $g_j(\xi v) = \xi^{d_j} g_j(v) = g_j(v)$ for all j. Thus $\xi v \in Wv$.

If $s \in W, v \in U(s, \xi)$ then $g_j(v) = g_j(sv) = g_j(\xi v) = \xi^{d_j} g_j(v)$. Thus if $j > r, g_j(v) = 0$. ■

Needed from algebraic geometry. To study the vartiety X_θ

Zariski topology on $\mathbb{C}^n$ the closed subsets are

$$\mathbb{C}^n(S) = \{v \in \mathbb{C}^n | f(v) = 0, f \in S\}$$

with $S \subset \mathcal{O}(\mathbb{C}^n)$. An affine variety is a Z-closed subset of some $\mathbb{C}^n$. We endow it with the subspace topology. If $X \subset \mathbb{C}^n$ is closed then

$$\mathcal{I}_X = \{f \in \mathcal{O}(\mathbb{C}^n) | f(X) = \{0\}\}$$

is an ideal in $\mathcal{O}(\mathbb{C}^n)$. The algebra $\mathcal{O}(X) = \{f_{|X} | f \in \mathcal{O}(\mathbb{C}^n)\}$ is isomorphic with $\mathcal{O}(\mathbb{C}^n)/\mathcal{I}_X$. If X and Y are affine varieties then a morphism $\Phi : X \to Y$ is a map satifying $\Phi^* \mathcal{O}(Y) \subset \mathcal{O}(X)$. This means that $\Phi = (f_1, ..., f_m)$, $f_i \in \mathcal{O}(\mathbb{C}^n)$.

Nullstellensatz 1. If $\mathcal{I}$ is an ideal in $\mathcal{O}(\mathbb{C}^n)$ such that $X_\mathcal{I} = \{x \in \mathbb{C}^n | \mathcal{I}(x) = 0\} = \emptyset$ then $X_\mathcal{I} = \mathcal{O}(\mathbb{C}^n)$.

Nullstellensatz 2, If $\mathcal{I}$ is an ideal in $\mathcal{O}(\mathbb{C}^n)$ then

$$\{f \in \mathcal{O}(\mathbb{C}^n) | f(X_\mathcal{I}) = 0\} =$$

$$\{f \in \mathcal{O}(\mathbb{C}^n) | f^k \in \mathcal{I} \text{ for some } k > 0\} := \sqrt{\mathcal{I}}.$$

Both are true for X and affine variety.

Two affine varieties, X, Y are isomorphic if there exists a bijective morphism $\Phi : X \to Y$ such that Φ^{-1} is a morphism. In other words if and only if the rings $\mathcal{O}(X)$ and $\mathcal{O}(Y)$ are isomorphic.

An affine variety, X, is said to be irreducible if $\mathcal{O}(X)$ is an integral domain. This means geometrically that if

$$X = X_1 \cup X_2$$

with X_i both closed then at least one is X.

If V is an n–dimensional vector space then V is irreducible.

If X is an affine variety then we can write X as a finite union of closed irreducible subvarieties. Furthermore if $X = X_1 \cup X_2 \cup ... \cup X_r$, X_i irreducible and if no X_i

is contained an X_j with $i \neq j$ then the decomposition is unique up to order. In this case the varieties are called the irreducible components of X.

Lemma. If X is an irreducible affine variety and if U_1 and U_2 are non-empty Zariski open subsets then $U_1 \cap U_2 \neq \emptyset$.

The dimension of an irreducible affine variety, X, is the transcendence degree of the quotient field of $\mathcal{O}(X)$. The dimension of X is the maximum of the dimensions of the irreducible components.

The dimension of $\mathbb{C}^n$ is n.

V a finite dimensional vector space. $\mathcal{O}_j(V)$ the subspace of polynomials of degree less than or equal to j. If $\mathcal{I}$ is an ideal in $\mathcal{O}(V)$ and $A = \mathcal{O}(V)/\mathcal{I}$ then set $A_j = \mathcal{O}_j(V) + \mathcal{I}$.

Lemma. If $\mathcal{I}$ is an ideal in $\mathcal{O}(V)$ and if $A = \mathcal{O}(V)/\mathcal{I}$ then there exists a polynomial $p_{\mathcal{I}}(t) \in \mathbb{Q}[t]$ such that if

j is sufficiently large that $\dim A_j = p_{\mathcal{I}}(j)$. This polynomial is called the Hilbert polynomial of $\{A_j\}$.

Proposition. If $\mathcal{I}$ is an ideal in $\mathcal{O}(V)$ and if $A = \mathcal{O}(V)/\mathcal{I}$ and if $X = \{x \in V | I(x) = 0\}$ then $\dim X = \deg p_I$.

If $\mathcal{I}$ is homogeneous then we have a grade on $A = \mathcal{O}(V)/\mathcal{I}$ by setting $A^j = \mathcal{O}^j(V) + \mathcal{I}$. Clearly $A_j = \oplus_{k \leq j} A^j$.

As before there is a polynomial $q_{\mathcal{I}}(t) \in \mathbb{Q}[t]$ such that $\dim A^j = q(j)$ for $j >> 1$

A Zariski closed subset, X, of V is called a cone if

$$x \in X \Longrightarrow zx \in X, z \in \mathbb{C}.$$

If $X \subset V$ is a cone then $\mathcal{I}_X = \{f \in \mathcal{O}(V) | f(X) = 0\}$ is homogeneuous. Thus $A = \mathcal{O}(V)/\mathcal{I}_X$ is graded.

We have a polynomial $q_X(t) \in \mathbb{Q}[t]$ such that

$$\dim A^j = q_X(j), j >> 1.$$

Proposition. $q_X(t) = \frac{r_X}{(d-1)!} t^{d-1} +$ lower order. $d = \dim X$.

The number r_X is called the degree, $\deg(X)$, of the cone X.

Lemma. If U is a subspace of V then $\deg(U) = 1$.

Proof. $\mathcal{I}_U = \mathcal{O}(V)U^\perp = \mathcal{O}(U)$ with $U^\perp = \{\lambda \in V^* | \lambda(U) = 0\}$. Since $\mathcal{O}(U)$ is isomorphic with $\mathbb{C}[x_1, ..., x_{\dim U}]$ and the nullstellensatz implies that $\mathcal{I}_U = \sqrt{\mathcal{O}(V)U^\perp}$ we see that $\mathcal{I}_U = \mathcal{O}(V)U^\perp$

If $\dim U = k$ then

$$\dim \mathcal{O}^j(U) = \binom{k+j-1}{k-1} = \frac{\prod_{i=1}^{k-1}(j+i)}{(k-1)!}.$$

So

$$q_X(t) = \frac{\prod_{i=1}^{k-1}(t+i)}{(k-1)!} = \frac{t^{k-1}}{(k-1)!} + \text{lower}.$$

■

Lemma Suppose that $X = X_1 \cup X_2$, X_i cone in V, $\dim X_i = d$ and let $Y = X_1 \cap X_2$. If $\dim Y < \dim X$ then $\deg X = \deg X_1 + \deg X_2$.

Proof. If $R = \mathcal{O}(\mathbb{C}^n)$ then we have (inclusion exclusion)

$$0 \to R/\mathcal{I}_X \to R/\mathcal{I}_{X_1} \oplus R/\mathcal{I}_{X_2} \to R/\left(\mathcal{I}_{X_1} + \mathcal{I}_{X_2}\right) \to 0$$

since $\mathcal{I}_{X_1} + \mathcal{I}_{X_2}$ defines Y this implies that the polynomial q which gives $\dim \left(R/\left(\mathcal{I}_{X_1} + \mathcal{I}_{X_2}\right)\right)^j$ is of lower degree than $d-1$. we see that and so $q_X(r) = q_{X_1}(r) +$

$q_{X_2}(r) - q_{\mathcal{I}_{X_1}+\mathcal{I}_{X_2}}(r)$ for r sufficiently large. So the leading term is

$$\frac{(\deg X_1 + \deg X_1)}{(d-1)!} t^{d-1}.$$

■

Proposition. Let $f_1, ..., f_n$ be a homogeneous system of parameters for $\mathbb{C}^n$. Let $A \subset \{1, ..., n\}$ be s subset. If $X_A = \{x \in \mathbb{C}^n | f_i(x) = 0, i \in A\}$ then every irreducible component of X_A has dimension $n - |A|$.

This is exactly Macaulay's theorem.