Lemma. If $W \subset GL(n, \mathbb{C})$ is a finite subgroup generated by complex reflections and if $f_1, \ldots, f_n$ is a set of homogeneous generators of $O(\mathbb{C}^n)^W$ then $f_1, \ldots, f_n$ is a system of parameters of $O(\mathbb{C}^n)$.

Proof. Suppose that $f_i(x) = 0$ for all $i = 1, \ldots, n$. Then $f(x) = 0$ for all $f \in O(\mathbb{C}^n)^W$. Thus $x \in W0 = 0$. □

This gives the more sophisticated proof of the freeness that was proved using Chevalley’s Lemma.

If $W$ is a finite subgroup of $GL(V)$ and $\zeta$ is a primitive $m$–th root of 1 and if $s \in W$ then set

$$U(s, \zeta) = \{v \in V | sv = \zeta v\}.$$  

An element $s \in W$ is said to be regular if there exists primitive $m$–th root of 1, $\zeta \neq 1$ such that $U(s, \zeta)$ contains a regular element.

If $U(s \zeta)$ contains a regular element and $\zeta$ is a primitive $m$–th root of 1 then $s^m = I$. 
We will now embark on the proof of the following theorem of Springer.

**Theorem** Let $W \subset GL(n, \mathbb{C})$ be a finite reflection group and let $\theta$ be a regular element of $W$ of order $m$. Let $g_1,\ldots,g_n$ be an algebraically independent set of homogeneous generators of $\mathcal{O}(\mathbb{C}^n)^W$ arranged so that

$$m \mid \text{deg } g_i, i = 1,\ldots,r, m \nmid \text{deg } g_i, i > r.$$ 

Let $\xi$ be a primitive $m$–th root of 1 and $a = U(\theta, \xi)$ then $W^\theta_a$ is a complex reflection group and if $f_i = g_i|a$ then

$$\mathcal{O}(a)^W_\theta = \mathbb{C}[f_1,\ldots,f_r].$$

$\theta$ a regular element of order $m$ and $\xi$ a primitive $m$–th root of unity such that $U(\theta, \xi)$ contains a regular element.

**Lemma** (1) If $s \in W$ and $U(\theta, \xi) \subset U(s, \xi)$ then $s = \theta$. 
(2) If \( s \in W \) and \( sU(\theta, \xi) \subset U(\theta, \xi) \) then \( s \in W^\theta \).

**Proof.** To prove (1) we note that \( s^{-1}\theta|_{U(\theta,\xi)} = I \) thus \( s = \theta \). For (2) note that \( sU(\theta, \xi) = U(s\theta s^{-1}, \xi) \). Thus (1) implies (2). ■

Let \( g_1, \ldots, g_n \) be homogeneous generators of \( \mathcal{O}(\mathbb{C}^n)^W \) we assume that \( m|\deg g_i, i = 1, \ldots, r \) and \( m \nmid \deg g_i, i > r \).

**Lemma** \( X_\theta = \cap_{i > r}\{v \in V|g_i(v) = 0\} = \cup_{s \in W}U(s, \xi) \).

**Proof.** Let \( v \in \cap_{i > r}\{v \in V|g_i(v) = 0\} \) then \( g_j(\xi v) = \xi^{d_j}g_j(v) = g_j(v) \) for all \( j \). Thus \( \xi v \in Wv \).

If \( s \in W, v \in U(s, \xi) \) then \( g_j(v) = g_j(sv) = g_j(\xi v) = \xi^{d_j}g_j(v) \). Thus if \( j > r, g_j(v) = 0 \). ■
Needed from algebraic geometry. To study the variety $X_\theta$

Zariski topology on $\mathbb{C}^n$ the closed subsets are

$$\mathbb{C}^n(S) = \{ v \in \mathbb{C}^n | f(v) = 0, f \in S \}$$

with $S \subset \mathcal{O}(\mathbb{C}^n)$. An affine variety is a $\mathbb{Z}$-closed subset of some $\mathbb{C}^n$. We endow it with the subspace topology. If $X \subset \mathbb{C}^n$ is closed then

$$\mathcal{I}_X = \{ f \in \mathcal{O}(\mathbb{C}^n) | f(X) = \{ 0 \} \}$$

is an ideal in $\mathcal{O}(\mathbb{C}^n)$. The algebra $\mathcal{O}(X) = \{ f|_X | f \in \mathcal{O}(\mathbb{C}^n) \}$ is isomorphic with $\mathcal{O}(\mathbb{C}^n)/\mathcal{I}_X$. If $X$ and $Y$ are affine varieties then a morphism $\Phi : X \to Y$ is a map satifying $\Phi^*\mathcal{O}(Y) \subset \mathcal{O}(X)$. This means that $\Phi = (f_1, \ldots, f_m), f_i \in \mathcal{O}(\mathbb{C}^n)$.

**Nullstellensatz 1.** If $\mathcal{I}$ is an ideal in $\mathcal{O}(\mathbb{C}^n)$ such that $X_{\mathcal{I}} = \{ x \in \mathbb{C}^n | \mathcal{I}(x) = 0 \} = \emptyset$ then $X_{\mathcal{I}} = \mathcal{O}(\mathbb{C}^n)$.

**Nullstellensatz 2.** If $\mathcal{I}$ is an ideal in $\mathcal{O}(\mathbb{C}^n)$ then

$$\{ f \in \mathcal{O}(\mathbb{C}^n) | f(X_{\mathcal{I}}) = 0 \} =$$
\[ \{ f \in \mathcal{O}(\mathbb{C}^n) | f^k \in \mathcal{I} \text{ for some } k > 0 \} := \sqrt{\mathcal{I}}. \]

Both are true for \( X \) and affine variety.

Two affine varieties, \( X, Y \) are isomorphic if there exists a bijective morphism \( \Phi : X \to Y \) such that \( \Phi^{-1} \) is a morphism. In other words if and only if the rings \( \mathcal{O}(X) \) and \( \mathcal{O}(Y) \) are isomorphic.

An affine variety, \( X \), is said to be irreducible if \( \mathcal{O}(X) \) is an integral domain. This means geometrically that if

\[ X = X_1 \cup X_2 \]

with \( X_i \) both closed then at least one is \( X \).

If \( V \) is an \( n \)-dimensional vector space then \( V \) is irreducible.

If \( X \) is an affine variety then we can write \( X \) as a finite union of closed irreducible subvarieties. Furthermore if \( X = X_1 \cup X_2 \cup \ldots \cup X_r, X_i \) irreducible and if no \( X_i \)
is contained an $X_j$ with $i \neq j$ then the decomposition is unique up to order. In this case the varieties are called the irreducible components of $X$.

**Lemma.** If $X$ is an irreducible affine variety and if $U_1$ and $U_2$ are non-empty Zariski open subsets then $U_1 \cap U_2 \neq \emptyset$.

The dimension of an irreducible affine variety, $X$, is the transcendence degree of the quotient field of $\mathcal{O}(X)$. The dimension of $X$ is the maximum of the dimensions of the irreducible components.

The dimension of $\mathbb{C}^n$ is $n$.

$V$ a finite dimensional vector space. $\mathcal{O}_j(V)$ the subspace of polynomials of degree less than or equal to $j$. If $\mathcal{I}$ is an ideal in $\mathcal{O}(V)$ and $A = \mathcal{O}(V)/\mathcal{I}$ then set $A_j = \mathcal{O}_j(V) + \mathcal{I}$.

**Lemma.** If $\mathcal{I}$ is an ideal in $\mathcal{O}(V)$ and if $A = \mathcal{O}(V)/\mathcal{I}$ then there exists a polynomial $p_{\mathcal{I}}(t) \in \mathbb{Q}[t]$ such that if
$j$ is sufficiently large that $\dim A_j = p_{\mathcal{I}}(j)$. This polynomial is called the Hilbert polynomial of $\{A_j\}$.

**Proposition.** If $\mathcal{I}$ is an ideal in $\mathcal{O}(V)$ and if $A = \mathcal{O}(V)/\mathcal{I}$ and if $X = \{x \in V | I(x) = 0\}$ then $\dim X = \deg p_{\mathcal{I}}$.

If $\mathcal{I}$ is homogeneous then we have a grade on $A = \mathcal{O}(V)/\mathcal{I}$ by setting $A^j = \mathcal{O}^j(V) + \mathcal{I}$. Clearly $A_j = \bigoplus_{k \leq j} A^j$.

As before there is a polynomial $q_{\mathcal{I}}(t) \in \mathbb{Q}[t]$ such that $\dim A^j = q(j)$ for $j \gg 1$

A Zariski closed subset, $X$, of $V$ is called a cone if

$$x \in X \implies zx \in X, z \in \mathbb{C}.$$
If $X \subset V$ is a cone then $\mathcal{I}_X = \{ f \in \mathcal{O}(V) | f(X) = 0 \}$ is homogeneous. Thus $A = \mathcal{O}(V)/\mathcal{I}_X$ is graded.

We have a polynomial $q_X(t) \in \mathbb{Q}[t]$ such that

$$\dim A^j = q_X(j), j >> 1.$$ 

**Proposition.** $q_X(t) = \frac{r_X}{(d-1)!} t^{d-1} + \text{lower order. } d = \dim X.$

The number $r_X$ is called the degree, $\deg(X)$, of the cone $X$.

**Lemma.** If $U$ is a subspace of $V$ then $\deg(U) = 1$.

**Proof.** $\mathcal{I}_U = \mathcal{O}(V) U^\perp = \mathcal{O}(U)$ with $U^\perp = \{ \lambda \in V^* | \lambda(U) = 0 \}$. Since $\mathcal{O}(U)$ is isomorphic with $\mathbb{C}[x_1, \ldots, x_{\dim U}]$ and the nullstellensatz implies that $\mathcal{I}_U = \sqrt{\mathcal{O}(V) U^\perp}$ we see that $\mathcal{I}_U = \mathcal{O}(V) U^\perp$.
If \( \dim U = k \) then
\[
\dim \mathcal{O}^j(U) = \binom{k + j - 1}{k - 1} = \frac{\prod_{i=1}^{k-1} (j + i)}{(k - 1)!}.
\]
So
\[
q_X(t) = \frac{\prod_{i=1}^{k-1} (t + i)}{(k - 1)!} = \frac{t^{k-1}}{(k - 1)!} + \text{lower}.
\]

Lemma Suppose that \( X = X_1 \cup X_2 \), \( X_i \) cone in \( V \), \( \dim X_i = d \) and let \( Y = X_1 \cap X_2 \). If \( \dim Y < \dim X \) then \( \deg X = \deg X_1 + \deg X_2 \).

Proof. If \( R = \mathcal{O}(\mathbb{C}^n) \) then we have (inclusion exclusion)
\[
0 \to R/\mathcal{I}_X \to R/\mathcal{I}_{X_1} \oplus R/\mathcal{I}_{X_2} \to R/\left(\mathcal{I}_{X_1} + \mathcal{I}_{X_2}\right) \to 0
\]
since \( \mathcal{I}_{X_1} + \mathcal{I}_{X_2} \) defines \( Y \) this implies that the polynomial \( q \) which gives \( \dim \left( R/\left(\mathcal{I}_{X_1} + \mathcal{I}_{X_2}\right) \right)^j \) is of lower degree than \( d - 1 \). we see that and so \( q_X(r) = q_{X_1}(r) + \)
\( qX_2(r) - q\mathcal{I}_{X_1} + \mathcal{I}_{X_2}(r) \) for \( r \) sufficiently large. So the leading term is

\[
\frac{(\deg X_1 + \deg X_1) t^{d-1}}{(d - 1)!}.
\]

\textbf{Proposition.} Let \( f_1, \ldots, f_n \) be a homogeneous system of parameters for \( \mathbb{C}^n \). Let \( A \subset \{1, \ldots, n\} \) be a subset. If \( X_A = \{ x \in \mathbb{C}^n | f_i(x) = 0, i \in A \} \) then every irreducible component of \( X_A \) has dimension \( n - |A| \).

This is exactly Macaulay’s theorem.