Let $W \subset GL(n, \mathbb{C})$ be a finite group generated by complex reflections.

W is said to be decomposable if we can write $\mathbb{C}^n$ as $V_1 \oplus V_2$ and $W = W_1 \times W_2$ with $W_i \subset GL(V_i)$ generated by complex reflections. It is called indecomposable if it is not decomposable. Shephard Todd show that this is the same as there are no non-zero proper invarriant subspaces i.e. irreducible. The table giving the classification of irreducble finite complex reflectiongroups (IFCRG) has 3 infinite families and 34 exceptionl examples.

Exercise. Using the Shephard Todd table show that if $n \geq 3$ and if W is in one of the infinite families of IFCRG Let $d_1 \leq d_2 \leq .. \leq d_n$ be the degrees of a homogeneous set of generators for the invariants. If $d_1 = 2$ then either d_2 or d_3 is 4.

If $v \in \mathbb{R}^n - \{0\}$ then we define a reflection about v to be

$$s_v w = w - \frac{2\langle v, w\rangle}{\langle v, v\rangle} v.$$

If $\lambda(w) = \langle v, w\rangle$ and $u = \frac{v}{\langle v,v\rangle}$ then $s_v = s_{\lambda,-1,u}$ on $\mathbb{C}^n$.

We look at the subgroup, W, of $GL(9,\mathbb{R}) \subset GL(9,\mathbb{C})$ generated by the reflections corresponding to the elements, R:

$$e_i - e_j, 1 \leq i < j \leq 9$$

and

$$e_i + e_j + e_k - \frac{1}{3}(e_1 + ... + e_9), 1 \leq i < j < k \leq 9.$$

The element $e_1 + ... + e_9$ is perpendicular to all of these elements.

Exercise. In $GL(\{v \in \mathbb{C}^9 | \sum v_i = 0\})$ W is irreducible.

Set $\Phi = R \cup -R$. The definition is correct.

Exercise. The reflections $s_u.u \in R$ permute the elements of Φ.

Lemma. If $\dim V < \infty$ and $G \subset GL(V)$ permutes a finite set $S \subset V$ that spans V then G is finite.

Proof. If $s \in G$ and $s_{|S} = I$. Then if $v \in V$ then $v = \sum_{u \in S} a_u u$. Thus $sv = v$. Thus $|G| \leq |S|!$. ■

This implies (applying the lemma to the orthogonal complement to $e_1 + ... + e_9$) yields $|W| < \infty$.

Thus $\sum x_i$ is an invariant. The tables imply that the set of invariants for W are generated by 8 more elements of degrees:

$$2, 8, 12, 14, 18, 20, 24, 30.$$

Since $n = 8$ it must be 37 on the list. We will show that there is a regular element, θ, of order 3. So the corresponing invariants for W^θ are of degrees

$$12, 18, 24, 30$$

By Springer's result. Thus W^θ is of order $12 * 18 * 24 * 30 = 155,520$. The group restricted to $(e_1 + ... + e_9)^\perp$ is number 32 in the Shephard-Todd list.

$$\mathbf{R}_1\colon \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \omega^2 & \\ & & & 1 \end{pmatrix}, \quad \mathbf{R}_2\colon \frac{-i}{\sqrt{3}} \begin{pmatrix} \omega & \omega^2 & \omega^2 & 0 \\ \omega^2 & \omega & \omega^2 & 0 \\ \omega^2 & \omega^2 & \omega & 0 \\ 0 & 0 & 0 & i\sqrt{3} \end{pmatrix}$$

$$\mathbf{R}_3\colon \begin{pmatrix} 1 & & & \\ & \omega^2 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \quad \mathbf{R}_4\colon \frac{-i}{\sqrt{3}} \begin{pmatrix} \omega & -\omega^2 & 0 & -\omega^2 \\ -\omega^2 & \omega & 0 & \omega^2 \\ 0 & 0 & i\sqrt{3} & 0 \\ -\omega^2 & \omega^2 & 0 & \omega \end{pmatrix}$$

In the preceding refection group there is another group generated by some of the reflections,

Here we take the zero sets for the three functionals

$$\lambda_1 = x_1 + x_2 + x_3, \lambda_2 = x_4 + x_5 + x_6, \lambda_3 + x_7 + x_8 + x_9.$$

We take the subgroup generated by the reflections about $u \in R$ with $\lambda_i(u) = 0$. The invariants of this group are $\lambda_1, \lambda_2, \lambda_3$ and a polynomail of degree 2 but none of degree 4 so it nust be 35 on the list

$$2, 5, 6, 8, 9, 12.$$

Restricting to the subspace

$$\lambda_1 = 0, \lambda_2 = 0, \lambda_3 = 0$$

We will see that there is a regular automorphism of order 3. Thus Springer's result implies that the corresponding invariants are of degrees:

$$6, 9, 12$$

According to the shephard to list this must be number 25.

The examples 35,36,37 with $n = 6, 7, 8$ are the Weyl groups E_6, E_7, E_8.

V an dimensional vector space over $\mathbb{C}$. Then a subgroup, G, of $GL(V)$ is called an affine algebraic group if it is the set of zeros of elements of $\mathcal{O}(End(V))$.

We endow $GL(V)$ with the structure of an affine variety by realizing considering it as

$$G = \{(X, t) \in End(V) \times \mathbb{C} | \, (\det X)\, t = 1\}.$$

Thus $\mathcal{O}(GL(V)) = \mathcal{O}(End(V))_{\mathsf{det}}$. Then is the functions of the form $f(X)\,\mathsf{det}(X)^{-k} \quad f \in \mathcal{O}(End(V))$. Thus an affine algebraic group is just Zariski closed subgroup of $GL(V)$.

In particular, G is a closed subgroup of $GL(V)$ in the topology it gets by putting a Hermitian inner product,$\langle , \rangle$ on V using the norms $\|v\| = \langle v, v\rangle^{\frac{1}{2}}$ for $v \in V$ and $\|X\| = (trX^*X)^{\frac{1}{2}}$ for $X \in End(V)$ to define metrics on V and $End(V)$ and therefore on $GL((V)$.

If G is a closed subgroup of $GL(V)$ then its Lie algebra, $Lie(G)$ is the set of $X \in End(V)$ such that

$$e^{tX} = \sum_{m=0}^{\infty} \frac{(tX)^m}{m!} \in G$$

for all $t \in \mathbb{R}$. This series converges absolutely since $\left\|X^k\right\| \leq \|X\|^k$. Then one has

1. $Lie(G)$ is a real subspace of $End(V)$.

2. If $X, Y \in Lie(G)$ then $[X, Y] = XY - YX \in Lie(G)$.

To prove 1 we use

$$\lim_{m \to \infty} \left(e^{\frac{X}{m}} e^{\frac{Y}{m}} \right)^m = e^{X+Y}$$

and

$$\lim_{m \to \infty} \left(e^{\frac{X}{m}} e^{\frac{Y}{m}} e^{-\frac{X}{m}} e^{-\frac{Y}{m}} \right)^{m^2} = e^{[X,Y]}.$$

Exercise. If you haven't seen this before then look at section 1.2 in Symmetry, Represntations and Invariants.

Lemma. If G is an affine algebraic subgroup of $GL(V)$ then $Lie(G)$ is a a complex subspace of $End(V)$.

Proof. G is the set of zeros of $S \subset \mathcal{O}(End(V))$. If $f \in S, X \in Lie(G)$ we have

$$f(e^{tX}) = 0$$

for all t in $\mathbb{R}$ but $z \longmapsto f(e^{zX})$ is holomorphic for $z \in \mathbb{C}$. Thus $f(e^{zX}) = 0$ all $z \in \mathbb{C}$. ■

Theorem. If G is a closed subgroup of $GL(V)$ then there exists an open neighborhood of 0 in $Lie(G)$, U_0, such then $\exp(U) = U_I$ is an open neighborhood of I in G in the metric topology and

$$\exp : U_0 \to U_1$$

is a homeomorphism.

Define $\log : U_I \to U_0$ to be the inverse map to $\exp_{|U_0}$.

$$G = \bigcup_{g \in G} gU_I.$$

If $g \in G$ then define $\Phi_g : gU_I \to U_o$ by $\Phi_g = \log \circ L_{g^{-1}}$ ($L_g x = gx$). If $gU_I \cap hU_I \neq \emptyset$ then the map

$$\Phi_h \circ \Phi_g^{-1} : \Phi_g(gU_I \cap hU_I) \to \Phi_h(gU_I \cap hU_I)$$

is analytic with analytic inverse.

The details for all of this material are left as exercises for those who have not seen them before.

Examples of affine algebraic groups

1. $GL(V), Lie(G) = End(V)$.

2. $SL(V) = \{g \in GL(V) | \det g = 1\}. Lie(G) = \{X \in End(V) | trX = 0\}$.

3. On V put $(.., ...)$ a non-degenerate symmetric bilinear form. Let

$O(V, (,)) = \{g \in GL(V) | (gv, gw) = (v, w), v, w \in V\}$.

and $SO(V, (,)) = O(V, (,)) \cap SL(V)$. $Lie(SO(V, (,))) = Lie(O(V, (,))) =$

$$\{X \in End(V) | (Xv, w) = -(v, Xw), v, w \in V\}.$$

4. On V put ω a non-degenerate alternating bilinear form $(\omega(v, w) = -\omega(w, v))$.

$Sp(V, (,)) = \{g \in GL(V) | \omega(gv, gw) = \omega(v, w), v, w \in V\}$.

$Lie(Sp(V, \omega)) = \{X \in End(V) | \omega(Xv, w) = -\omega(v, Xw), v, v$