Let $W \subset GL(n, \mathbb{C})$ be a finite group generated by complex reflections.

$W$ is said to be decomposable if we can write $\mathbb{C}^n$ as $V_1 \oplus V_2$ and $W = W_1 \times W_2$ with $W_i \subset GL(V_i)$ generated by complex reflections. It is called indecomposable if it is not decomposable. Shephard Todd show that this is the same as there are no non-zero proper invariant subspaces i.e. irreducible. The table giving the classification of irreducible finite complex reflectiongroups (IFCRG) has 3 infinite families and 34 exceptional examples.

**Exercise.** Using the Shephard Todd table show that if $n \geq 3$ and if $W$ is in one of the infinite families of IFCRG Let $d_1 \leq d_2 \leq \ldots \leq d_n$ be the degrees of a homogeneous set of generators for the invariants. If $d_1 = 2$ then either $d_2$ or $d_3$ is 4.

If $v \in \mathbb{R}^n - \{0\}$ then we define a reflection about $v$ to be

$$s_vw = w - \frac{2\langle v, w \rangle}{\langle v, v \rangle}v.$$
If $\lambda(w) = \langle v, w \rangle$ and $u = \frac{v}{\langle v, v \rangle}$ then $s_v = s_{\lambda,-1,u}$ on $\mathbb{C}^n$.

We look at the subgroup, $W$, of $GL(9, \mathbb{R}) \subset GL(9, \mathbb{C})$ generated by the reflections corresponding to the elements, $R$:

$$e_i - e_j, 1 \leq i < j \leq 9$$

and

$$e_i + e_j + e_k - \frac{1}{3}(e_1 + \ldots + e_9), 1 \leq i < j < k \leq 9.$$ 

The element $e_1 + \ldots + e_9$ is perpendicular to all of these elements.

**Exercise.** In $GL(\{v \in \mathbb{C}^9| \sum v_i = 0\})$ $W$ is irreducible.

Set $\Phi = R \cup -R$. The definition is correct.

**Exercise.** The reflections $s_{u,u} \in R$ permute the elements of $\Phi$. 
Lemma. If \( \dim V < \infty \) and \( G \subset GL(V) \) permutes a finite set \( S \subset V \) that spans \( V \) then \( G \) is finite.

Proof. If \( s \in G \) and \( s|_S = I \). Then if \( v \in V \) then 
\[
v = \sum_{u \in S} a_u u.
\]
Thus \( sv = v \). Thus \( |G| \leq |S|! \).  

This implies (applying the lemma to the orthogonal complement to \( e_1 + \ldots + e_9 \)) yields \( |W| < \infty \).

Thus \( \sum x_i \) is an invariant. The tables imply that the set of invariants for \( W \) are generated by 8 more elements of degrees:

\[
2, 8, 12, 14, 18, 20, 24, 30.
\]

Since \( n = 8 \) it must be 37 on the list. We will show that there is a regular element, \( \theta \), of order 3. So the corresponding invariants for \( W^\theta \) are of degrees

\[
12, 18, 24, 30
\]

By Springer’s result. Thus \( W^\theta \) is of order \( 12 \times 18 \times 24 \times 30 = 155,520 \). The group restricted to \( (e_1 + \ldots + e_9)^\perp \) is number 32 in the Shephard-Todd list.
In the preceding reflection group there is another group generated by some of the reflections,

Here we take the zero sets for the three functionals

\[ \lambda_1 = x_1 + x_2 + x_3, \lambda_2 = x_4 + x_5 + x_6, \lambda_3 = x_7 + x_8 + x_9. \]

We take the subgroup generated by the reflections about \( u \in R \) with \( \lambda_i(u) = 0 \). The invariants of this group are \( \lambda_1, \lambda_2, \lambda_3 \) and a polynomial of degree 2 but none of degree 4 so it must be 35 on the list

\[ 2, 5, 6, 8, 9, 12. \]
Restricting to the subspace

\[ \lambda_1 = 0, \lambda_2 = 0, \lambda_3 = 0 \]

We will see that there is a regular automorphism of order 3. Thus Springer’s result implies that the corresponding invariants are of degrees:

\[ 6, 9, 12 \]

According to the shephard to list this must be number 25.

The examples 35, 36, 37 with \( n = 6, 7, 8 \) are the Weyl groups \( E_6, E_7, E_8 \).

\( V \) an dimensional vector space over \( \mathbb{C} \). Then a subgroup, \( G \), of \( GL(V) \) is called an affine algebraic group if it is the set of zeros of elements of \( \mathcal{O}(End(V)) \).

We endow \( GL(V) \) with the structure of an affine variety by realizing considering it as

\[ G = \{(X, t) \in End(V) \times \mathbb{C} | (\det X) t = 1 \}. \]
Thus $O(GL(V)) = O(End(V))_{\text{det}}$. Then is the functions of the form $f(X) \det(X)^{-k} \quad f \in O(End(V))$. Thus an affine algebraic group is just Zariski closed subgroup of $GL(V)$.

In particular, $G$ is a closed subgroup of $GL(V)$ in the topology it gets by putting a Hermitian inner product, $\langle , \rangle$ on $V$ using the norms $\|v\| = \langle v, v \rangle^{\frac{1}{2}}$ for $v \in V$ and $\|X\| = (\text{tr}X^*X)^{\frac{1}{2}}$ for $X \in End(V)$ to define metrics on $V$ and $End(V)$ and therefore on $GL((V))$.

If $G$ is a closed subgroup of $GL(V)$ then its Lie algebra, $Lie(G)$ is the set of $X \in End(V)$ such that

$$e^{tX} = \sum_{m=0}^{\infty} \frac{(tX)^m}{m!} \in G$$

for all $t \in \mathbb{R}$. This series converges absolutely since $\|X^k\| \leq \|X\|^k$. Then one has

1. $Lie(G)$ is a real subspace of $End(V)$.
2. If $X, Y \in \text{Lie}(G)$ then $[X, Y] = XY - YX \in \text{Lie}(G)$.

To prove 1 we use

$$\lim_{m \to \infty} \left( \frac{X}{e^m} \frac{Y}{e^m} \right)^m = e^{X+Y}$$

and

$$\lim_{m \to \infty} \left( \frac{X}{e^m} \frac{Y}{e^m} - \frac{X}{e^m} \frac{Y}{e^m} \right)^m = e^{[X,Y]}.$$

**Exercise.** If you haven’t seen this before then look at section 1.2 in Symmetry, Representations and Invariants.

**Lemma.** If $G$ is an affine algebraic subgroup of $GL(V)$ then $\text{Lie}(G)$ is a a complex subspace of $\text{End}(V)$.

**Proof.** $G$ is the set of zeros of $S \subset \mathcal{O}(\text{End}(V))$. If $f \in S, X \in \text{Lie}(G)$ we have

$$f(e^{tX}) = 0$$
for all $t$ in $\mathbb{R}$ but $z \mapsto f(e^{tX})$ is holomorphic for $z \in \mathbb{C}$. Thus $f(e^{zX}) = 0$ all $z \in \mathbb{C}$. ■

**Theorem.** If $G$ is a closed subgroup of $GL(V)$ then there exists an open neighborhood of 0 in $Lie(G)$, $U_0$, such then $\exp(U) = U_I$ is an open neighborhood of $I$ in $G$ in the metric topology and

$$\exp : U_0 \to U_1$$

is a homeomorphism.

Define $\log : U_I \to U_0$ to be the inverse map to $\exp|U_0$.

$$G = \bigcup_{g \in G} gU_I.$$ If $g \in G$ then define $\Phi_g : gU_I \to U_0$ by $\Phi_g = \log \circ L_{g^{-1}} (L_g x = gx)$. If $gU_I \cap hU_I \neq \emptyset$ then the map

$$\Phi_h \circ \Phi_g^{-1} : \Phi_g(gU_I \cap hU_I) \to \Phi_h(gU_I \cap hU_I)$$

is analytic with analytic inverse.

The details for all of this material are left as exercises for those who have not seen them before.
Examples of affine algebraic groups

1. $GL(V)$, $Lie(G) = End(V)$.

2. $SL(V) = \{ g \in GL(V) | \det g = 1 \}. Lie(G) = \{ X \in End(V) | trX = 0 \}$.

3. On $V$ put $(\ldots, \ldots)$ a non-degenerate symmetric bilinear form. Let

$$O(V, (,)) = \{ g \in GL(V) | (gv, gw) = (v, w), v, w \in V \}.$$ 

and $SO(V, (,)) = O(V, (,)) \cap SL(V)$. $Lie(SO(V, (,))) = Lie(O(V, (,))) =$

$$\{ X \in End(V) | (Xv, w) = -(v, Xw), v, w \in V \}.$$

4. On $V$ put $\omega$ a non-degenerate alternating bilinear form $(\omega(v, w) = -\omega(w, v))$.

$$Sp(V, (,)) = \{ g \in GL(V) | \omega(gv, gw) = \omega(v, w), v, w \in V \}.$$ 

$Lie(Sp(V, \omega)) = \{ X \in End(V) | \omega(Xv, w) = -\omega(v, Xw), v, w \in V \}$.