Examples of compact subgroups

1. $U = U(V, \langle , \rangle) = \{u \in GL(V) | u^* = u^{-1}\}, Lie(U) = \{X \in End(V) | u^* = u^{-1}\}$.

2. $SU(V, \langle , \rangle) = \{u \in U | \det u = 1\}$, $Lie(SU) = \{X \in Lie(U) | trX = 0\}$.

3. $V = \mathbb{C}^n$ if $(z, w) = \sum z_i w_i$ then $O(n, \mathbb{C}) = O(\mathbb{C}^n, (,))$ and $SO(n, \mathbb{C}) = SO(\mathbb{C}^n, (,))$.The corresponding compact groups are $O(n) = O(n, \mathbb{C}) \cap GL(n, \mathbb{R})$ and $SO(n) = SO(n, \mathbb{C}) \cap GL(n, \mathbb{R})$. Lie $SO(n) = Lie(O(n)) = \{X \in End(\mathbb{R}^n) | (Xv, w) = -(v, Xw), v, w \in V\}$

For the fourth example we look at the quaternions $\mathbb{H}$. This is the four dimensional algebra over $\mathbb{R}$ with identity 1 and three more basis vectors i, j, k with multiplication $ij = k$,$ik = -j$, $jk = i$ and $i^2 = -1$. It can also be looked upon the real subalgebra of $M_2(\mathbb{C})$

$$\mathbb{H} = \left\{ h(a,b) = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} | a, b \in \mathbb{C} \right\}.$$

$$1 \to I, i \to h(0,1), j \to h(0,i), k \to h(i,0).$$

If $z \in \mathbb{H}$, $z = a+bi+cj+dk$ we define $\bar{z} = a-bi-cj-dk$. We note that if $z \to h(a,b)$ then $\bar{z} \to h(a,b)^*$. We define for $X \in M_n(\mathbb{H})$, $X^* = \bar{X}^T$ here the "bar" means bar every matrix entry.

4. $Sp(n) = \{X \in M_n(\mathbb{H}) | XX^* = I\}$. We map $M_n(\mathbb{H})$ into $M_{2n}(\mathbb{C})$ by writing $X = M_{2n}(\mathbb{C})$ as and $n \times n$ of 2×2 blocks and in the r, s block we put $h(a_{rs}, b_{rs})$. Then in this form the adjoints are consistant. Thus $M_n(\mathbb{H}) \cap SU(2n) = Sp(n)$.

If we use the form

$$\omega(x,y) = x_1y_2 - x_2y_1 + x_3y_4 - x_4y_3 + ... + x_{2n-1}y_{2n} - x_{2n}y_{2n-1}$$

then $Sp(n) = Sp(\mathbb{C}^{2n}, \omega) \cap U(2n)$.

Proposition. If G is an algebraic group then the Z–connected components are the irreducible components.

Proof. Let S be the set of all g that are in at least two irreducible components. Then since g acts morphically $GS = S$. Thus either S is empty or $S = G$. If $X_1, ..., X_m$ are the irreducible components of G containing the identity then the closure, Y, of $X_1 \cdots X_m$ is irreducible and contains the identity. Hence $m = 1$ and $Y = X_1$.Thus $S = \emptyset$. This implies that if X and Y are irreducible components then $X \cap Y = \emptyset$. Thus the irreducible components are the connected components. ■

An algebraic subgroup, G, of $GL(V)$ is said to be reductive if there exists a Hermitian inner product $\langle ..., ... \rangle$ on V such that if $g \in G$ then $g^* \in G$.

See GIT for alternative definitions of reductive.

Exercise. All of the algebraic examples above are reductive.

If G is reductive and if $X \in Lie(G)$ then $X^* \in Lie(G)$ since $\left(e^{tX}\right)^* = e^{tX^*}$. Set $B(X, Y) = trXY$ for $X, Y \in Lie(G)$. Then B is non-degenerate ($B(X, X^*) >$ 0 if $X \neq$ 0).

If K is a compact group then there is a Borel measure, μ, on K, with the following properties:

1. If f is continuous then f is integrable and if and $f(x) \geq$ 0 for all $x \in G$ and $f \neq$ 0 then

$$\int_K f d\mu > 0.$$

2. If f is continuous then $\int_K L_k^* f d\mu = \int_K R_k^* f d\mu = \int_K f d\mu$.

3. $\mu(K) = 1$.

This measure is unique and we will write it as

$$\int_K f(k) dk.$$

Lemma. If K is a compact group and $(\rho.V)$ is a finite dimensional continuous representation of K then there exists an inner product, $\langle , \rangle$, on V such that $\rho(K)$ is contained in the unitary group of $(V, \langle , \rangle)$(i.e. $\langle , \rangle$ is K-invariant.

Proof. Let $(,)$ be an inner product on V. Set

$$\langle v, w \rangle = \int_K (\rho(k)v, \rho(k)w)\, dk.$$

Then 1. implies that if $v \neq 0$ then $\langle v, v \rangle > 0$. ■

Theorem. If $K \subset GL(n, \mathbb{C})$ is a compact subgroup then the Zariski closure of K is a reductive subgroup.

Proof. Since K is compact there is an inner product on $\mathbb{C}^n$, $\langle , \rangle$, such that K is contianed in the unitary group of $\langle , \rangle$. We may assume that $\langle , \rangle$ is the usual inner product. If $f \in \mathcal{O}(M_n(\mathbb{C}))$ vanishes on K then

$$g(X) = \overline{f(X^*)}$$

is a polynomial that vanishes in K. Thus if $g \in GL(n, \mathbb{C})$ is in the Z-closure of K then $g^* \in G$. We note that the Z-closure of a group in $GL(n, \mathbb{C})$ is a group and if G is a Z-closed subgroup invariant under adjoint then $G \cap U(n)$ is maximal compact in G. ■

Lemma. If $H \subset GL(n, \mathbb{C})$ is a subgroup then its Z-closure is a sugroup.

Proof. Let $f \in \mathcal{O}(M_n(\mathbb{C}))$ and let $h \in H$ then $L_h^* f(X) = f(hX)$ and $R_h^* f(X) = f(Xh)$ are polynomials that vanish on H. Thus if G is the Z-closure of H then $HG \subset G$ but then if f vanishes on $R_g^* f$ vaishes on H for $g \in G$. Hence $GG \subset G$. ■

Lemma. Let $f \in \mathcal{O}(M_n(\mathbb{C}))$. If $X \in M_n(\mathbb{C})$ satisfies $X^* = X$ and if $f(e^{nX}) = 0$ for all $n \in \mathbb{Z}$ then $f(e^{tX}) = 0$ for all $t \in \mathbb{R}$.

Proof. We may assume that X is diagonal with real diagonal entries $\lambda_1, ..., \lambda_n$. Thus

$$f(e^{tX}) = \sum a_J e^{t \sum j_i \lambda_i} = \sum_{k=1}^{r} c_k e^{t\mu_k}$$

with $\mu_k \in \mathbb{R}$ and distinct. We may assume $\mu_1 < ... < \mu_r$ then

$$\lim_{t \to \infty +} e^{-t\mu_r} f(e^{tX}) = \lim_{t \to \infty +} \sum_{k=1}^{r} c_k e^{t(\mu_k - \mu_r)} = c_r.$$

But $f(e^{nX}) = 0$ for all $n \in \mathbb{Z}$ so $c_r = 0$. ■

Theorem. If G is a Zariski closed subgroup of $GL(n, \mathbb{C})$ such that if $g \in G$ then $g^* \in G$ and if $K = U(n) \cap G$ then

$$K \times iLie(K) \to G$$

$$k, X \longmapsto ke^X$$

is a homeomorphism and K is a maximal compact subgroup of G and G is the Zariski closure of K.

Proof. Let $g \in G$ then $gg^* = A$ defines a positive definite element of G. If we diagonalize A using an orthonormal basis then we see that $A = e^X$ with $X^* = X$. Thus $e^{nX} = A^n \in G$ thus $e^{tX} \in G$ for all $t \in \mathbb{R}$.Hence $X \in Lie(G)$. Setting $A^{\frac{1}{2}} = e^{\frac{1}{2}X}$ then $(A^{\frac{1}{2}})^2 = A$. Thus

$$gA^{-\frac{1}{2}}\left(gA^{-\frac{1}{2}}\right)^* = gA^{-1}g^* = g\left(g^{-1}\left(g^*\right)^{-1}\right)g^* = I.$$

Set $\mathfrak{p} = \{X \in Lie(G) | X^* = X\}$. Then we have shown that the map

$$K \times \mathfrak{p} \to G$$

$$u, X \longmapsto ue^X$$

is surjective. We note that $\mathfrak{p} = iLie(U(n) \cap Lie(G)$. Also the map is injective since if $u, v \in G \cap U(n)$ and $X, Y \in$

$$g = ue^X = ve^Y$$

then $g^*g = e^{2X} = e^{2Y}$ so $X = Y$ hence $u = v$. The map sends closed sets to closed sets. Indeed, If Z is closed in $G \cap U(n) \times \mathfrak{p}$ and $z_j \in Z$ is a sequence such that if $z_j = (u_j, X_j)$ then if $u_j e^{X_j}$ coverges to $u_o e^{X_o}$ then

$$e^{2X_j} = \left(u_j e^{X_j}\right)^* u_j e^{X_j} \to \left(u_o e^{X_o}\right)^* u_o e^{X_o} = e^{2X_o}.$$

So $X_j \to X_o$ hence $k_j \to k_o$. Thus its image is closed. This implies that the inverse map is continuous.

If K_1 is a compact subgroup in G containing K and $u \in K_1$ then $u = ke^X$ with $X \in \mathfrak{p}$ and $k \in K$. Thus $e^X \in K_1$. Since the eigen values of X are real this is only possible if $X = 0$ (Exercise).

Let $\phi \in \mathcal{O}(GL(n,\mathbb{C}))$ be such that $\phi_{|K} = 0$. Then $\phi(ke^{tX}) = 0$ for $k \in K, X \in \mathsf{Lie}(K)$ and $t \in \mathbb{R}$. This function of t as the restriction of a holomorphic function on $\mathbb{C}$ so this vanishing is also true for $t \in \mathbb{C}$. Thus $\phi(ke^{Z}) = 0$ for $k \in K$ and $Z \in i\mathsf{Lie}(K)$.Hence G is contained in the Z–closure of K. Since G is Z–closed G is the Z–closure. ■