Note Title 10/18/2010

$V \subset \bar{V} \subset U \subset \mathbb{R}^n$ open V open

$\bar{V}$ compact.

$\sigma : [a,b] \to U$

piecewise smooth $\qquad \sigma(a) = p, \ \sigma(b) = q$

$L(\sigma) \geq Cr$ with C depending on $\bar{V}$.

$$L(\sigma) = \int \left(\langle A(\sigma(t))\sigma'(t), \sigma'(t) \rangle \right)^{1/2} dt$$

$A : U \to M_n(\mathbb{R})$ $\quad C^\infty \quad (A_{ij}(x) \in C^\infty)$

$\overline{A(x)} = A(x)$ $\qquad \langle A(x)v, v \rangle > 0, \ x \in U, v \neq 0.$

Observe: If $v \in S^{n-1}$, $x \notin \overline{V}$

then $C_1^2 \leq \langle A(x)v, v \rangle \leq C_2^2$

$C_1, C_2 > 0$. Since $\overline{V} \times S^{n-1}$ is compact.

$v \neq 0$, $v = \|v\| \cdot \frac{v}{\|v\|}$

Conclude if $x \notin \overline{V}$ then

$$C_1^2 \|v\|^2 \leq \langle A(x)v, v \rangle \leq C_2^2 \|v\|^2$$

We prove assertion with $C = C_1$.

The assertion if $\exists a \leq s \leq b$ such that $\sigma(s) \notin \overline{B_p(r)}$ then $L(\sigma) \geq C_1 r$.

Let $t_0 = \inf \{ t \mid \sigma(t) \notin \overline{B_p(r)} \}$. $0 < t_0 \leq s$

$\|p - \sigma(t_0)\| = r.$ $\qquad \sigma([a,t_0]) \subset B_p(r)$

$$\int_a^b \langle A(\sigma(t))\sigma'(t), \sigma'(t)\rangle^{1/2} dt$$

$$\geq \int_a^{t_0} \langle A(\sigma(t))\sigma'(t), \sigma'(t)\rangle^{1/2} dt$$

$$\geq C_1 \int_a^{t_0} \langle \sigma'(t), \sigma'(t)\rangle^{1/2} dt$$

$$\geq C_1 \|p - \sigma(t_0)\| = C_1 r.$$

If M is C^∞ manifold with a Riemannian structure g.

Assume if M is connected then

if $p, q \in M$ $\exists$ $\sigma : [a,b] \to M$

piecewise smooth such that

$\sigma(a) = p$, $\sigma(b) = q$.

$$L(\sigma) = \int_a^b g_{\sigma(t)}(\sigma'(t), \sigma'(t))\, dt$$

$$\sigma'(t) = d\sigma\left(\frac{d}{dt}_t\right) \in T_{\sigma(t)}(M).$$

↑ define $d(p,q) = \inf_{\sigma} L(\sigma)$, σ piecewise smooth $\sigma(a) = p$, $\sigma(b) = q$.

We assume M is connected

If $p \neq q$ then $d(p,q) > 0$.

Let (U, Ψ) be a chart with $p \in U$. $y_1, \ldots, y_n$ local coord on U.

$$g_{ij}(m) = g_m\left(\frac{\partial}{\partial y_{i_m}}, \frac{\partial}{\partial y_{j_m}}\right).$$

On $\Psi(U) = U'$ have $A(x) =$

$$\left[g_{ij}\left(\Psi^{-1}(x)\right) \right] \quad x \in U'.$$

piecewise C^∞

If $\sigma : [a,b] \to U$ is C^∞

then $\tilde{\sigma} = \Psi \circ \sigma : [a,b] \to U'$ piecewise C^∞

$$A(x) = \left[g_{ij}(\Phi^{-1}(x)) \right], \quad x \in U'$$

$$L(\sigma) = \int_a^b \langle A(\tilde{\gamma}(t))\,\tilde{\gamma}'(t), \tilde{\gamma}'(t) \rangle^{1/2}\, dt$$

$\sigma : [a,b] \to U$.

Now we can prove $d(p,q) > 0$.

Call σ a curve $q \notin U$ and $\gamma : [a,b] \to M$ $\sigma(a) = p$, $\sigma(b) = q$, $p \in U$, we can assume we have

$$\overline{B_{\Phi(p)}(r)} \subset V \subset \bar{V} \subset U' \quad U' = \Phi(U)$$

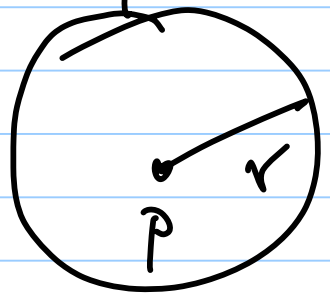
$\bar{V}$ compact. V open. $\exists a \le t_1 \le b$ such that $\Phi(\sigma([a,t_1])) \subset \overline{B_{\Phi(p)}(r)}$. $\forall r > 0$

$\tilde{\sigma}([a,t_1]) =$

$\Rightarrow L(\sigma) \geq C_1 r \quad C_1 > 0.$

If $p, q \in U$. But $\exists a \leq b$ such that $\sigma(s) \notin U$ the same argument implies $L(\sigma) \geq C_1 r. > 0$

May assume $\sigma([a,b]) \subset \bar{U}$



if and $q \notin B_p(r)$ then $L(\sigma) \geq C_1 r.$

If ~~$\sigma([a,b])$~~ $\sigma : [a,b] \to B_p(r)$.

$L(\sigma) \geq C_1 \| \tilde{\sigma}(a) - \tilde{\sigma}(b) \| > 0.$

$(p \neq q)$.

We claim that d gives the topology on M. Consider $p \in M$. (U, Ψ) chart. on U, g defines a Riemannian structure. $d|_{U \times U}$ is the Riemannian metric on U. $U' = \Psi(U)$

$\overline{B_p(r)} \subset V \subset \bar{V} \subset U'$ assume $\Psi = I$, $U = U'$

Consider $x, y \in B_p(\frac{r}{4})$ assume

We show

$$c_2 \|x - y\| \leq d(x, y) \leq c_1 \|x - y\|$$

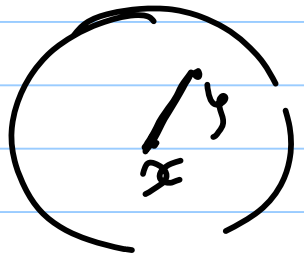
Note that $B_x(\|x-y\|) \subset B_p(r)$

$\because$ $\|x-z\| < \|x-y\| \le \frac{1}{2}r$

Then $\|z-p\| \le r$.

$$\|z-p\| \le \|x-z\| + \|x-p\| \le \frac{3}{4}r.$$

$\Rightarrow$ lower estimate.



$$\sigma(t) = (1-t)x + ty$$

$$\sigma'(t) = x - y.$$

M is connected then

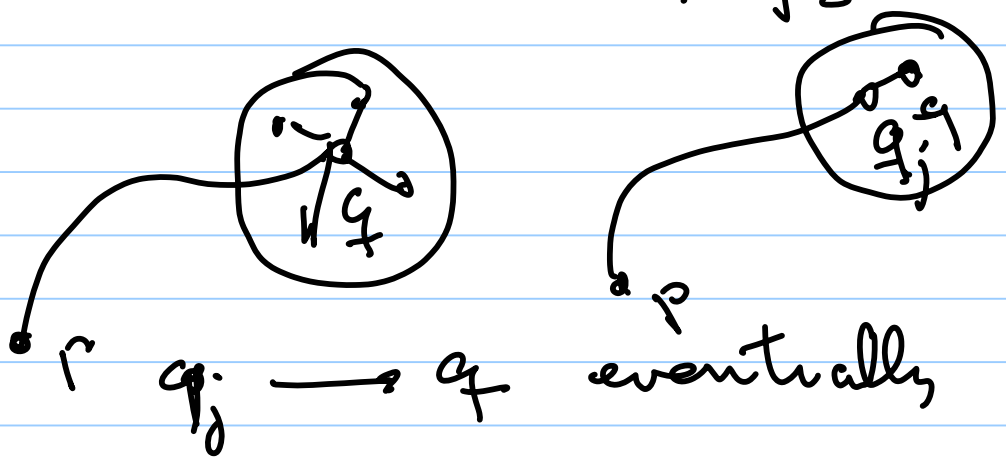
if $p, q \in M$ $\exists$ $\sigma : [a,b] \to M$

piecewise smooth such that

$\sigma(a) = p$, $\sigma(b) = q$. (i.e. we can join q to p)

Fix $p \in M$. Let $N = \{ q \mid$ we can join q to $p \}$.

$N \supset \{p\}$.

N open:

N closed