If \( M \) satisfies the second axiom of countability, then \( M \) has a Riemannian structure if and only if it is connected and metrizable via the Riemannian metric.

Theorem. If \( M \) is a metrizable connected manifold, then \( M \) satisfies the second axiom of countability (i.e., \( M \) has a countable base for its topology.)
The following are equivalent for a connected $C^\infty$ manifold.

1. 2.nd axim
2. Has a Riemannian structure
3. Metrizable

$M$ $C^\infty$ manifold and $N = (X, \mathcal{B})$ another $C^\infty$ manifold. Then $N$ is called a submanifold of $X \subset M$. $X \subset M$ is a subset.
\[ i_x : x = x \]

1. If \( i_x : N \rightarrow M \) then
   \[ i_x \circ c = \text{id} \quad \text{and} \quad (d i_x)_{p} : T_p(N) \rightarrow T_p(M) \]
   is \( \text{injective} \) for all \( p \in N \).

Examples. (1) \( S^1 \times S^1 = T^2 \)

\[ T^2 = \{ (z, w) \mid |z| = 1, |w| = 1 \} \subset \mathbb{C}^2 = \mathbb{R}^4 \]

\[ \mathbb{R} \rightarrow T^2, \quad t \mapsto (e^{2 \pi it}, e^{2 \pi it}) \]

\[ f(\mathbb{R}) = X \subset T^2, \quad f \text{ is \textit{injective}.} \]
up to topology on X making

t a homeomorphism.

\( (X, f') \) is an atlas.

\[ \tau : \mathbb{N} \rightarrow \mathbb{T}^2 \]

\( \tau \) and \( (d\tau_x)_* \) are injective.

(Details are an exercise)

2. \( M = \mathbb{R}^2 \), \( X = \mathbb{R}^2 \)

\[ X = \bigcup_{x \in \mathbb{R}} (x, \mathbb{R}) \] , topologize

as a disjoint union of \( (x, \mathbb{R}) \) \( \cong \mathbb{R} \).
\(\{x, 1R\}_{x_2}^\pi_2 \mid (x, 1R)\} \) define a \( C^0 \) atlas.

\( \dot{x} \) is \( C^0 \) and \((d_1 x)(x, t)\) is injective.
Let $f : N \to M$ be a Riemannian manifold, $(M, \langle \cdot, \cdot \rangle)$ be a Riemannian structure.

If $f$ is $C^\infty$ and $df_p : T_p(N) \to T_{f(p)}(M)$ is injective for all $p \in N$, then $f$ is called an immersion.

Define $f^* \langle \cdot, \cdot \rangle$ a Riemannian structure on $N$ by $f^* \langle v, w \rangle_p = \langle df_p v, df_p w \rangle_{f(p)}$. 
\[
\Rightarrow \text{ if an immersion exists of a connected manifold } N \text{ into a Riemannian manifold then } N \text{ satisfies the second axiorn.}
\]

\[
\Rightarrow \text{ if } N \subset M \text{ is a submanifold of a } \mathbb{C} \text{- manifold and if } N \text{ is connected then } N \text{ satisfies the second axiom...}
\]
Proof of 2nd axiom result is essentially the one in Kostrikin–Neretin Volume 1 Appendix 2.

Steps 1. If \((X, d)\) is a metric space then \((X, d)\) satisfies the 2nd axiom \(\iff (X, d)\) has a countable dense set (separability).

Step 2. Set up an equivalence class \(x \sim y\) such that \([x]\) equivalence class is open and separable.
Step 1. 2nd axiom \( \Rightarrow \) Sep.

\[ \forall U \text{ countable basis for } X, \exists \text{ some } p \in D \]

\[ U = \bigcup_{j=1}^{\infty} U_j, \quad D = \{ p_j : p_j \text{ some element of } U_j \} \]

\( \{ p_j \} \) is dense in \( X \).

\( p \in X, p \in U \subset X, U \text{ open } \Rightarrow U_j \subset U \)

and so \( p_j \in U \).

Sep \( = \) 2nd axiom \( \Rightarrow \) D \( \forall r > 0 \), countable dense \( \{ B_p(\frac{1}{n}) : p \in D, r \text{ rational} \} \)
Ex. Show this is a basis for the topology.

M metrizable connected manifold. Take $d$ a metric on its $\Omega$. We show that it is separable. $x, y \in M$

$x \sim y$ is the relation in $y \in B_x(r)$ and $x \in B_y(r')$ with $B_x(r), B_y(r')$ separable.

$x \sim y \iff y \sim x$
Also \( xRx \).

Define \( \lambda A \subseteq M \) such that \( xRx \) if there exists \( y \in A \) such that \( xRy \).

\( \lambda A \subseteq M \)

Define \( SA \) by \( S\lambda A = SA \)

\( S^{n+1}A = S(S^nA) \).

\( x \sim y \) if there exists \( n \) such that \( y \in S^n x = S^n\{x\} \).

This is an equivalence relation.
New steps.

(a) \( \forall x \in M \text{ then } S(x) \text{ is open.} \)

(b) \( \forall x \in M \text{ is removable then so is in } S \).

(c) \[ [x] \text{ (rel. to } n) \Rightarrow [x] = \bigcup_{n \geq 1} S^n x \]
\[ S^n x \subseteq S^{n+1} x. \]

will prove the theorem.