Note Title 10/20/2010

If M satisfies the second axiom of countability then M has a Riemannian structure $\Rightarrow$ If M is connected then M is metrizable via the Riemannian metric.

<u>Theorem.</u> If M is a metrizable connected C^∞ manifold then M satisfies the second axiom of countability (i.e. M has a countable basis for its topology.

Cor. The following are equivalent for a connected C^∞ manifold.

(1) 2nd axiom

(2) Has a Riemannian structure

(3) Metrizable

M C^∞ manifold and $N = (X, \mathcal{B})$ another C^∞ manifold then N is called a submanifold if $X \subset M$ is a subset.

$(i_X(x) = x)$

II. If $i_X : N \longrightarrow M$ then

i_X is C^∞ and

$$(di_X)_p : T_p(N) \longrightarrow T_p(M)$$

is injective for all $p \in N$.

<u>Examples</u>. (1) $S^1 \times S^1 = T^2$

$$T^2 = \{(z,w) \mid |z| = |w| = 1\} \subset \mathbb{C}^2 = \mathbb{R}^4.$$

$$\mathbb{R} \longrightarrow T^2, \quad t \overset{f}{\longmapsto} (e^{2\pi i t}, e^{2\pi i \sqrt{2} t})$$

$f(\mathbb{R}) = X \subset T^2$, f is injective.

use topology on X making f a homeomorphism.

$\therefore (X, f^{-1})$ is an C^∞ atlas, giving N

$$i_X : N \longrightarrow T^2$$

C^∞ and $(di_X)_p$ is injective.

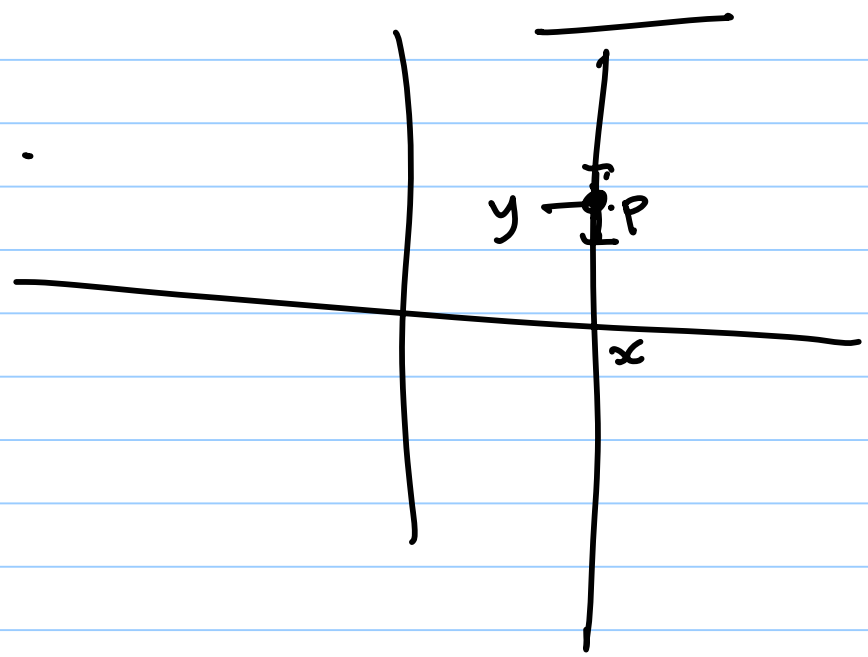
(Details are an exercise)

2. $M = \mathbb{R}^2$, $X = \mathbb{R}^2$

$$X = \bigcup_{x \in \mathbb{R}} (x, \mathbb{R}),$$ topologize as a disjoint union of $(x, \mathbb{R}) \underset{p_2}{\cong} \mathbb{R}$.

$\left\{ (x, \mathbb{R}) \xrightarrow{\pi_2} \Big|_{(x,\mathbb{R})} \right\}$ defines a C^∞ atlas.

i_x is C^∞ and $(di_x)_{(x,t)}$ is injective.



$\frac{\partial}{\partial y}_{(x,y)}$

N C^∞ manifold, $(M, \langle , \rangle)$

Let $f : N \longrightarrow M$ (i.e. $\langle , \rangle$ a Riemannian structure)

is C^∞ and

$$df_p : T_p(N) \longrightarrow T_{f(p)}(M)$$

is injective for all $p \in N$.

(Such a map is called an immersion).

define $f^*\langle \rangle$ a Riemannian structure on N by $f^*\langle v, w\rangle_p = \langle df_p v, df_p w\rangle_{f(p)}$.

$\Rightarrow$ if an immersion exists of a connected manifold N into a Riemannian manifold then N satisfies the second axiom.

$\Rightarrow$ if $N \subset M$ is a submanifold of a 2nd manifold and if N is connected then N satisfies the second axiom.

Proof of 2nd axiom result is essentially the one in Kobayashi-Nomizu Volume 1 Appendix 2.

<u>Step 1.</u> If (X, d) is a metric space then (X, d) satisfies the 2nd axiom $\Longleftrightarrow$ (X, d) has a countable dense set (separability)

<u>Step 2.</u> Set up an equivalence class $x \sim y$ such that $[x]$ equivalence class is open and seperable.

<u>Step 1.</u> 2nd axiom $\Rightarrow$ sep.

$\mathcal{U}$ countable basis for the top.

$\mathcal{U} = \{U_j\}_{j=1}^{\infty}$. $D = \{p_j \mid p_j$ some element of $\{U_j\}\}$.

$\{p_j\}$ is dense in X.

$p \in X$, $p \in U \subset X$, U open $\exists\, U_j \subset U$ and so $p_j \in U$.

Sep $\Rightarrow$ 2nd axiom. D countable dense $\{B_p(r) \mid p \in D, r > 0, r$ rational$\}$

Ex. Show this is a basis for the topology.

M metrizable connected manifold. Take d a metric for its top. We show that d is separable. $x, y \in M$

xRy is the assertion $y \in B_x(r)$ and $x \in B_y(r')$ with $B_x(r)$, $B_y(r')$ separable.

$xRy \iff yRx$

also xRx.

Define $A \subset M$, $SA = \{x \in M \mid \exists\, y \in A$ such that xRy is true$\}$

$A \subset SA$.

Define $S^n A$ by $S^1 A = SA$

$S^{n+1} A = S(S^n A)$.

$x \sim y$ if there exists n such that $y \in S^n x := S^n\{x\}$.

This is an equivalence relation.

New steps.

(a) If $x \in M$ then Sx is open.

(b) If $A \subset M$ is separable then so is SA.

(c) $[x]$ (rel. to $\sim$) $[x] = \bigcup_{n \geq 1} S^n x$

$S^n x \subset S^{n+1} x$.

will prove the theorem.