$X \in \mathrm{Vect}(M)$, $X_p \neq 0$.

Note Title 11/5/2010

<u>Lemma</u>. There exists $(U, \Psi)$ a chart for $M$ with $p \in U$ and $X|_U = \frac{\partial}{\partial x_j}$ where $x_1, \dots, x_m$ are the corresponding local coord in $U$.

<u>Proof</u>. Let $(V, \Phi)$ be a cubic chart around $p$ (i.e. $\Phi(V) = C_a^m$, $a > 0$

$$C_a^m = \{x \mid |x_i| < a, \ i = 1, \dots, m\}$$

$\Phi(p) = 0$) such that if $y_1, \dots, y_m$ are the corresponding local coordinates

Then $X_p = \sum_i a_i \frac{\partial}{\partial y_i}\Big|_p$ with $a_n \neq 0$.

and also $\varphi_t^X : V \to M$ exists

for $|t| < \varepsilon$. ($\varepsilon > 0$)

$$H(t, z) = \Phi(\varphi_t(\Phi^{-1}(z)))$$

$z \in C_a^m$. $\quad z \in C_a^{m-1}$

$$G(t, z) = H(t, (z, 0)).$$

$$G : (-\varepsilon, \varepsilon) \times C_a^{m-1} \longrightarrow \mathbb{R}^m.$$

$dG_{0,0}\left(\frac{\partial}{\partial t}_{0,0}\right) = d\Phi_p(X_p)$, $dG_{0,0}(e_j) =$

for $j = 1, \ldots, m-1$. $(\varphi_0 = I)$ $e_j$

Hence $dG_{0,0}$ is bijective.

$\exists$ $b>0$ $b<\varepsilon$, $b<a$ such that

$G : C_b^m \longrightarrow \underline{G(C_b^m)}$ open is a $X$,

diffeomorphism. $W$

$(W, G^{-1})$ has property that

it gives local coord $z_1, \dots, z_m$ such

$\frac{\partial}{\partial z_1} = d\Phi_{-}(X_{-})$ pull back

to $M$.

Want the following:

(1) $p \longmapsto V_p$ with $V_p \subset T_p(M)$ a vector subspace.

(2) assume $\dim V_p = n$ for all $p \in M$.

(3) $\forall$ $p \in M$ $\exists$, $U$ open, $p \in U$ and $X_1, \ldots, X_n \in \mathrm{Vect}(U)$ such that

$$\underline{V_q = \mathrm{span}_{\mathbb{R}}(X_{1_q}, \ldots, X_{n_q})} \text{ for all } q \in U.$$

We call this an $n$-dimensional subbundle of the tangent bundle.

If $p \in M$ does there exist an $n$-dimensional connected submanifold $N$, with $p \in N$ and

$$\mathrm{di}_q(T_q N) = V_q \quad \text{for all } q \in N. \quad \Psi|_W)$$

Assume that $N$ exists for $p$.

Then $\exists$ $(U, \Psi)$ cubic chart at $p$. $(\Psi(U) = C_a^m)$ such that $\Psi^{-1}(\underbrace{C_a^n}_{W} \times \overset{m-n}{0})$ is a chart for $N$. $p \in W$

$$v \in V_q, \quad q \in W, \quad v = \sum_{i=1}^{n} a_i \frac{\partial}{\partial y_i}\Big|_q$$

If $X_1, \ldots, X_n$ are as in (3) for $U$ (possibly having shrunk $U$)

Then $X_i\big|_U = \sum_{j=1}^{n} a_{ji} \frac{\partial}{\partial y_j}$.

Now calculate $[X_i, X_k]$

$x = y$

$$\sum_{l \le n,\ j \le n} a_{ji}\left(\frac{\partial}{\partial x_j} a_{lk}\right)\frac{\partial}{\partial x_l} - \sum_{l} a_{lk}\left(\frac{\partial}{\partial x_l} a_{ji}\right)\frac{\partial}{\partial x_j}$$

$$[X_i, X_j]_q \in V_q.$$

Same argument if $X, Y \in \mathrm{Vect}(M)$ are such that $X_q, Y_q \in V_q$ for all $q \in M$ then $[X, Y]_q \in V_q$ for all $q \in M$. This condition is called integrability i.e. $q \to V_q$ is is called an integrable subbundle of $T(M)$ if cond. satisfied.

(Warner called this a "distribution")

A submanifold $N$ of $M$ is called an integral submanifold for $V$ if for each $p \in N$, $T_p(N) = V_p$.

<u>Theorem</u>. Given $V$ an integrable subbundle of $T(M)$ of dim $n$, and $p \in M$ there exists a cubic chart $(U, \Psi)$ for $M$ at $p$ such that the slice

$$y_{n+1}(q) = 0, \ldots, y_m(q) = 0 \text{ in } U$$

is an integral submanifold for $V$.

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Make $M$ into an integral submanifold of $V$.

On $M$ we put a new topology defined as follows: If $p \in M$ let $(U, \Psi)$ be a n-dim slice through $p$ such the slice is an integral cubic chart for $p$ in $M$. We take Sub for a subbasis of the topology at $p$ many the open subsets of the slice. We use all such slices through $p$.

We must prove that $M$ with this proposed basis for a topology is a topological space.

(1) $\phi$, $M$ open obvious.

(2) a union of open sets is open.

(3) intersection of a finite number of open sets is open.