Manifolds with boundary.

\[ \mathbb{R}^n_{>0} = \{ (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n > 0 \} \]

Subspace topology:

If \( U \subset \mathbb{R}^n_{>0} \) is open and \( f : U \to \mathbb{R} \) is continuous, then we say that \( f \) is \( C^\infty \) if there exists \( f_p \in \mathbb{R}^n \) open such that \( p \in V \) and \( f \) extends to a \( C^\infty \) function on \( V \).

\( \mathbb{R}^2_{>0} \) as usual.
If \( U \subset \mathbb{R}^m \) is open and \( f: U \to \mathbb{R}^m \)
then \( f \) is said to be \( C^0 \) if
\[
f = (f_1, \ldots, f_m) : f_i \in C^0.
\]

If \( X \) is a Hausdorff topological
space then an \( \eta \)-chart on \( X \) with boundary
is a pair \((U, \tilde{\varphi}), \varphi: U \to \mathbb{R}^n \), \( \tilde{\varphi}(U) \) open
and \( \tilde{\varphi}: U \to \tilde{\varphi}(U) \) is a homeomorphism.

If \((U, \tilde{\varphi}), (V, \varphi)\) are charts then
the \( \psi \) are \( C^0 \) related if \( \psi(\tilde{\varphi}^{-1}(U \cap V)) \to \tilde{\varphi}(V) \nabla \tilde{\psi} = \tilde{\varphi}(U \cap V) \nabla \varphi \in C^0. \)
A $C^\infty$ structure with boundary is a collection $\{\mathcal{U}_x, \phi_x\}_{x \in \partial M}$ of charts that are pairwise $C^\infty$ related.

A $C^\infty$ structure with boundary is a maximal $C^\infty$ atlas with boundary.

$$\mathcal{B}^n = \{ x \in \mathbb{R}^n \mid \|x\| \leq 1 \}.$$
Interior of $M$ a manifold with boundary is the set of all points such that there exists a chart $(U, \Phi)$ where \( \Phi(U) \) is open in $\mathbb{R}^n - M^0$. $M^0$ clearly a $C^\infty$ $n$-manifold, and we set $\partial M = M - M^0$ closed.

\underline{Differential forms.} $M$ $n$-dim $C^\infty$ manifold.

$p \mapsto \omega \in \Lambda^r(T^*M)^*$ is $C^\infty$ as a map from $M \rightarrow T^*(M)$. $\Omega^r(M)$ denotes the space of all diff. $r$-forms.
\( \omega, \gamma \in \mathfrak{D}^r(M) \), \( p \to \omega_p + \eta_p \) defines an element of \( \mathfrak{D}^r(M) \). \( f \in \mathcal{C}^\infty(M) \)
\( p \mapsto f(p) \omega_p \in \mathfrak{D}^r(M) \). \( \forall X_1, \ldots, X_r \in \text{Vect}(M) \)
\( p \to \omega_p (X_1_p, \ldots, X_r_p) \) is \( \mathcal{C}^\infty \) on \( M \).
\( \omega(X_1, \ldots, X_r) = \text{sym}(\omega)(X_1, \ldots, X_r) \).

and it is \( \mathcal{C}^\infty(M) \) - multilinear.

Conversely if \( \omega : \text{Vect}(M) \times \ldots \times \text{Vect}(M) \to \mathcal{C}^\infty(M) \) is alternating and \( \mathcal{C}^\infty(M) \) - multilinear then it is given by an element of \( \mathfrak{D}^r(C^\infty(M)) \).

Reason: Need to know \( \mathcal{D} \)
\( X_1, \ldots, X_r, Y_1, \ldots, Y_r \in \text{Vect}(M) \).
and $X_i p = Y_i p, \ i = 1, \ldots, r$. Then

$$\omega(X_1, \ldots, X_r)(p) = \omega(Y_1, \ldots, Y_r)(p).$$

First observe that if $(U, \bar{Y})$ is a chart

then if $\varphi \in \mathcal{C}^\infty(M)$ is $\varphi \equiv 1$ in a neighborhood $p \in \bar{U}$ and $\bar{V} \subset U$, $\varphi \equiv 0$ on $M - U$.

Then

$$\omega(\varphi X_1, X_2, \ldots, X_r) = \varphi \omega(X_1, \ldots, X_r).$$

Can replace $X_1$ by $\varphi X_1 = \varphi \sum a_i \frac{\partial}{\partial x_i}$, $a_i \in \mathcal{C}^\infty(U)$.

If $\alpha_i(p) = 0$, $\omega(\varphi \sum a_i \frac{\partial}{\partial x_i}, X_2, \ldots, X_r)$

$$= \varphi(p) \sum \alpha_i(p) \omega(\frac{\partial}{\partial x_i}, \ldots) (p) = 0$$

conclude is

if $X_1 p = 0$ then $\omega(X_1, \ldots, X_r)(p) = 0$. 


Implies: \[ \omega(x_1, \ldots, x_r)(\theta) = \omega(\eta_1, x_2, \ldots, \eta_r, \theta), \]

\[ \omega \in \mathcal{D}(M), \quad \eta \in \mathcal{D}(M). \]

\[
\omega \wedge \eta = \sum_{\sigma \in S_{r+s}} \text{sgn}(\sigma) \omega(x_{\sigma(1)}, \ldots, x_{\sigma(r)}) \eta(x_{\sigma(r+1)}, \ldots, x_{\sigma(r+s)})
\]

We will define \( d: \mathcal{D}(M) \to \mathcal{D}^r(M) \)

\text{exterior derivative.}

If \( f: M \to N \) is \( C^\infty \) then \( \omega \in \mathcal{D}^r(M) \) we define \( f^* \omega \in \mathcal{D}^{r+r}(M) \).
\((f^*\omega)_p(v_1, \ldots, v_r) = \omega(df_p(v_1), \ldots, df_p(v_r))\)
\(v_1, \ldots, v_r \in T_p(M)\)

\[(1)\quad d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^r \omega \wedge d\eta\]
\(\omega \in \Omega^r(M), \eta \in \Omega^q(M)\).

\[(2)\quad df^*\omega = f^*d\omega.\]

\[(3)\quad d^2 = 0, \quad Z_{dR}^r(M)\]

\[(3) = 1 \text{ Ker } d \big|_{\Omega^r(M)} \subset d\Omega^{r-1}(M) = B_{dR}^r(M)\]

\(H_{dR}^r(M) = Z_{dR}^r(M)/B_{dR}^r(M)\).
(2) If \( \gamma: M \to N \) is \( C^\infty \) we have an induced map \( f^*: H^r_{\partial_\nu}(N) \to H^r_{\partial_\nu}(M) \)

\( \text{(1)} \quad \eta = \partial_\nu \omega \in Z^r_{\partial_\nu}(M), \eta \in Z^s_{\partial_\nu}(M) \)

Then \( w \wedge \eta \in Z^{r+s}_{\partial_\nu}(M) \) if \( w \in B^r_{\partial_\nu}, \eta \in Z^s_{\partial_\nu} \) then \( w \wedge \eta \in B^{r+s}_{\partial_\nu} \).

Have a product

\( H^r \times H^s \to H^{r+s} \) (cup product)