Note Title 12/1/2010

Manifolds with boundary.

$$\mathbb{R}^n_{\geq 0} = \{ (x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n \geq 0 \}.$$

subspace topology.

If $U \subset \mathbb{R}^n_{\geq 0}$ is open and $f : U \to \mathbb{R}$ is continuous then we say that f is C^∞ if there exists for each $p \in U$ $V \subset \mathbb{R}^n$ open such $p \in V$ and $f|_{V \cap U}$ extends to a C^∞ function on V.

C^∞ as usual

$\mathbb{R}^1_{\geq 0}$

If $U \subset \mathbb{R}^n_{\geq 0}$ is open and $f : U \to \mathbb{R}^m$ then f is said to be C^∞ if $f = (f_1, \dots, f_m)$, f_i C^∞.

If X is a Hausdorff topological space then an n-chart for X with boundary is a pair (U, Ψ), $\Psi : U \to \mathbb{R}^n_{\geq 0}$, $\Psi(U)$ open and $\Psi : U \to \Psi(U)$ is a homeomorphism.

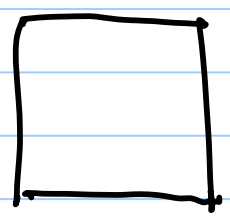
If (U, Ψ), (V, Φ) are n-charts then they are C^∞ related if $\Psi \circ \Phi^{-1} : \Phi(U \cap V) \to \mathbb{R}^n$ is C^∞ and $\Phi \circ \Psi^{-1} : \Psi(U \cap V) \to \mathbb{R}^n$ C^∞.

C^∞ atlas with boundary is is a collection $\{(U_\alpha, \Psi_\alpha)\}_{\alpha \in I}$ of charts with bd. that are pairwise C^∞ related.

A C^∞ structure with boundary is a maximal C^∞ atlas with boundary.

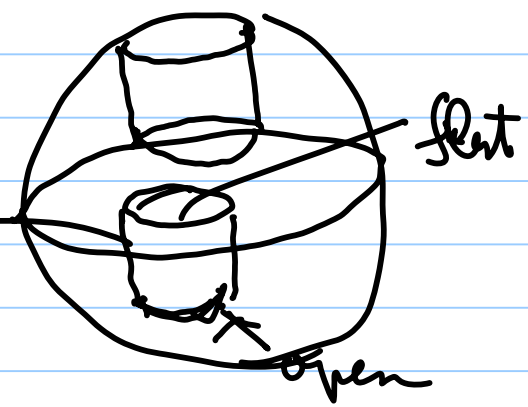
$(I, \mathbb{R}^n_{\geq 0})$ $\qquad$ $\mathbb{R}^n_{>0}$ $\qquad$ $\mathbb{R}^{n-1} \times (0)$

$\overline{B^n} = \{ x \in \mathbb{R}^n \mid \|x\| \leq 1 \}$.



is not

$\mathbb{R}^3_{\leq 0}$



Interior of M a n-manifold with boundary is the set of all points such that there exists a chart (U, Ψ) where $\Psi(U)$ is open in $\mathbb{R}^n_{\geq 0}$. M^o

M^o clearly a C^∞ n-manifold.

and we set $\partial M = M - M^o$ closed.

<u>Differential forms</u>. M m-dim C^∞ manif.

$p \mapsto \omega_p \in \Lambda^r T(M)^*$ is C^∞ differential p r-form

as a map from $M \longrightarrow T^*(M)$.

$\mathcal{A}^r(M)$ denotes the space of all diff. r-forms

$\omega, \eta \in \mathcal{D}^r(M)$, $p \longrightarrow \omega_p + \eta_p$ defines an element of $\mathcal{D}^r(M)$. $f \in C^\infty(M)$

$p \mapsto f(p)\omega_p \in \mathcal{D}^r(M)$. If $X_1, \ldots, X_r \in \mathrm{Vect}(M)$

$p \longmapsto \omega_p(X_{1p}, \ldots, X_{rp})$ is C^∞ on M.

$\omega(X_{\sigma 1}, \ldots, X_{\sigma r}) = \mathrm{sgn}(\sigma)\, \omega(X_1, \ldots, X_r)$.

and it is $C^\infty(M)$-multilinear.

Conversely if $\omega : \mathrm{Vect}(M) \times \cdots \times \mathrm{Vect}(M) \to C^\infty(M)$ is alternating and $C^\infty(M)$-multilinear then it is given by an element of $\mathcal{D}^r(M)$.

<u>Reason.</u> Need to know that if $X_1, \ldots, X_r, Y_1, \ldots, Y_r \in \mathrm{Vect}(M)$

and $X_{i_p} = Y_{i_p}$ $i = 1, \dots, r$. Then

$$\omega(X_1, \dots, X_r)(p) = \omega(Y_1, \dots, Y_r)(p).$$

First observe that if (U, Ψ) is a chart with coord $x_1, \dots, x_m$

Then if $\varphi \in C^\infty(M)$ is $\varphi \equiv 1$ in a nbd of $p \in U$ and $\overline{V} \subset U$, $\varphi \equiv 0$ on $M - U$.

Then $\omega(\varphi X_1, X_2, \dots, X_r) = \varphi \omega(X_1, \dots, X_r)$.

Can replace X_1 by $\varphi X_1 \big|_V = \varphi \sum_1^m a_i \frac{\partial}{\partial x_i}$, $a_i \in C^\infty(U)$

If $a_i(p) = 0$, $\omega\left(\varphi \sum_1^m a_i \frac{\partial}{\partial x_i}, X_2, \dots, X_r\right)$

$$= \varphi(p) \sum_1^m a_i(p) \omega\left(\frac{\partial}{\partial x_i}, \dots\right)(p) = 0$$

conclude is if $X_{1_p} = 0$ then $\omega(X_1, \dots, X_r)(p) = 0$

Implies: $\omega(X_1, \dots, X_r)(p) =$

$\omega(Y_1, X_2, \dots, X_r)(p)$,

$\omega \in \mathcal{D}^r(M)$, $\eta \in \mathcal{D}^s(M)$.

$$\omega \wedge \eta(X_1, \dots, X_{r+s}) = \frac{1}{(r+s)!} \sum_{\sigma \in S_{r+s}} \operatorname{sgn}(\sigma)\, \omega(X_{\sigma(1)}, \dots, X_{\sigma(r)})\, \eta(X_{\sigma(r+1)}, \dots, X_{\sigma(r+s)})$$

symmetric group

We will define $d : \mathcal{D}^r(M) \longrightarrow \mathcal{D}^{r+1}(M)$ exterior derivative.

If $f : M \longrightarrow N$ is C^∞ then if $\omega \in \mathcal{D}^r(N)$ we define $f^*\omega \in \mathcal{D}^r(M)$

by $(f^*\omega)_p(v_1,\dots,v_r) = \omega_{f(p)}(df_p(v_1),\dots,df_p(v_r))$

$v_1,\dots,v_r \in T_p(M)$

(1) $d(\omega\wedge\eta) = d\omega\wedge\eta + (-1)^r\omega\wedge d\eta$

$\omega\in\mathcal{D}^r(M)$, $\eta\in\mathcal{D}^s(M)$.

(2) $d f^*\omega = f^* d\omega$.

(3) $d^2 = 0$.

(3) $\Rightarrow$ $\underbrace{\operatorname{Ker} d|_{\mathcal{D}^r(M)}}_{= Z^r_{dR}(M)} \supset d\mathcal{D}^{r-1}(M) := B^r_{dR}(M)$

$H^r_{dR}(M) = Z^r_{dR}(M)/B^r_{dR}(M)$.

(2) If $f: M \longrightarrow N$ is C^∞ we have an induced map f^*, $H^r_{dr}(N) \to H^r_{dr}(M)$

(1) $\Rightarrow$ If $\omega \in Z^r_{dR}(M)$, $\eta \in Z^s_{dR}(M)$

then $\omega \wedge \eta \in Z^{r+s}_{dr}(M)$ if

$\omega \in B^r_{dr}$, $\eta \in Z^s_{dr}$ then

$\omega \wedge \eta \in B^{r+s}_{dR}$. Hence a product

$H^r \times H^s \longrightarrow H^{r+s}$ (cup product)