6 The solution to the quantum harmonic oscillator

We maintain the notation of the previous section. We are studying the quantum Harmonic oscillator

\[ H = -\frac{\hbar^2}{2m} \frac{d^2}{dq^2} + \frac{kq^2}{2} \]

with \( \omega = \sqrt{\frac{k}{m}} \). We have found an orthonormal basis of \( L^2(\mathbb{R}) \), \( \psi_n, n = 0, 1, ... \) with \( \psi_n \in \mathcal{S}(\mathbb{R}) \) and

\[ H \psi_n = \hbar \omega (n + \frac{1}{2}) \psi_n. \]

**Exercise.** The domain of \( \mathcal{H} \) is the space of all \( f \in L^2(\mathbb{R}) \) with

\[ \sum \left| \langle \psi_n | f \rangle \right|^2 (\frac{1}{2} + n)^2 < \infty. \]

(Hint use the Riesz representation theorem.) Use this to see that the closure is self adjoint.

We can thus form the solution to the Schrödinger equation with initial condition \( \phi(0) \)

\[ \phi(t) = e^{-\frac{i}{\hbar}Ht} \phi(0) = \sum_{n \geq 0} e^{-it\omega(\frac{1}{2} + n)} \langle \psi_n | \phi(0) \rangle \psi_n. \]

Notice that the phases that come into the quantum solution are essentially the same as those in the classical solution.

7 Unitary representations of \( \mathbb{R} \)

Let \( \mathcal{H} \) be a Hilbert space and let \( \text{End}(\mathcal{H}) \) be the space of all bounded operators on \( \mathcal{H} \). The strong operator topology on \( \text{End}(\mathcal{H}) \) is defined by the seminorms \( \mu_v(A) = \|Av\| \) for all \( v \in \mathcal{H} \).

If \( G \) is a topological group then a unitary representation of \( G \) on a Hilbert space \( \mathcal{H} \) is a homomorphism, \( U \), of \( G \) to \( U(\mathcal{H}) \) that is continuous in the strong operator topology. This is the same as saying that the map \( G \times \mathcal{H} \to \mathcal{H} \) given by \( g, v \mapsto U(g)v \) is continuous. To see this we note that \( \|U(g)v - U(g_0)v_0\| = \|(U(g) - U(g_0))v_0 + U(g)(v - v_0)\| \leq \mu_{v_0}(U(g) - U(g_0)) + \|v - v_0\| \).
Lemma 1 Let $\mathcal{D}$ be a dense subspace of $\mathcal{H}$ such that the maps $G \to \mathcal{H}$ given by $g \mapsto U(g)v$ are continuous for $v \in \mathcal{D}$. Then $U : G \to U(\mathcal{H})$ is strongly continuous.

Proof. Let $v_o \in \mathcal{D}$ and $v \in \mathcal{H}$ then

$$\mu_v(U(g) - U(g_o)) = \|U(g)v - U(g_o)v\| =$$

$$\|U(g)v_o - U(g_o)v_o + U(g)(v - v_o) + U(g_o)(v_o - v)\| \leq \mu_{v_o}(U(g) - U(g_o)) + 2\|v - v_o\|.$$ 

In this section we will give a classification of unitary representations of $\mathbb{R}$. We first note that we have shown in section 3 that if $H$ is a self adjoint operator on the Hilbert space $\mathcal{H}$ and if $P$ is the corresponding spectral measure. Then if $v$ is in the domain of $\overline{H}$

$$v \mapsto \int_{\mathbb{R}} e^{it\lambda} dP(\lambda)v$$

extends to a unitary representation of $\mathbb{R}$. We prove that the converse is also true.

Let $(U, \mathcal{H})$ be a unitary representation of $\mathbb{R}$. We define $\mathcal{H}^\infty$ to be the space of all $v \in \mathcal{H}$ such that the map $t \mapsto U(t)v$ defines a $C^\infty$ map of $\mathbb{R}$ to $\mathcal{H}$. We note that $\mathcal{H}^\infty$ is $U(\mathbb{R})$ invariant. We define an operator $L : \mathcal{H}^\infty \to \mathcal{H}^\infty$ by $Lv = \frac{d}{dt}_{t=0} U(t)v$. On $\mathcal{H}^\infty$ we define the seminorms

$$\psi = \xi_k(v) = \|L^k v\|.$$ 

Lemma 2 $\mathcal{H}^\infty$ endowed with these seminorms is a Fréchet space.

Proof. We need only show that it is complete. Let $\{v_j\}$ be a Cauchy sequence in $\mathcal{H}^\infty$ then using $\xi_0$ and $\xi_1$ we see that there exists $w \in \mathcal{H}$ such that $\lim_{j \to \infty} v_j = w$ in $\mathcal{H}$ and $\lim_{j \to \infty} Lv_j = w_1$ in $\mathcal{H}$. Now

$$\frac{d}{dt} U(t)v_j = U(t)Lv_j.$$ 

This implies

$$U(t)v_j = \int_0^t U(s)Lv_j ds - v_j.$$ 

2
taking the limit as $j \to \infty$ we have

$$U(t)w = \int_0^t U(s)w_1ds - w.$$  

This implies that $t \mapsto U(t)w$ is of class $C^1$ and $\frac{d}{dt}U(t)w = U(t)w_1$. We can now apply the same argument to $w_1$ using $\xi_1$ and $\xi_2$ to see that the map is $C^2$ and the second derivative is given by $\lim_{t \to \infty} L^2 v_j$. The result follows from the obvious inductive argument. ■

We note that if $v, w \in H^\infty$ then

$$0 = \frac{d}{dt} \left( U(t)v|U(t)w \right)_{|t=0} = \langle Lv|w \rangle + \langle v|Lw \rangle.$$  

Thus $iL$ is symmetric with domain $H^\infty$.

**Lemma 3** $iL$ is self adjoint.

**Proof.** Let $\widetilde{H}^\infty$ be the set of all $v \in H$ such the map $w \mapsto \langle v|iLw \rangle$ on $H^\infty$ extends to a continuous map of $H$ to $\mathbb{C}$. We note that if $w \in \widetilde{H}^\infty$ and if $v \in H^\infty$ then

$$\langle \int_0^t U(s)Lwds - w \rangle = \int_0^t \langle U(s)Lw|v \rangle ds - \langle w|v \rangle =$$

$$\int_0^t \langle Lw|U(s^{-1})v \rangle ds - \langle w|v \rangle = -\int_0^t \langle w|U(s^{-1})Lv \rangle ds - \langle w|v \rangle$$

$$= \langle w|U(t^{-1})v \rangle = \langle U(t)v|w \rangle.$$  

Since $H^\infty$ is dense we see that $\int_0^t U(s)Lwds - w = U(t)w$ for $w \in \widetilde{H}^\infty$. We therefore have shown that if $w \in \widetilde{H}^\infty$ then $U(t)w$ is of class $C^1$. If $w \in H$ is such that $t \mapsto U(t)w$ is of class $C^1$ then we define $Sw = \frac{d}{dt}U(t)w_{|t=0}$ then if $v \in H^\infty$

$$\langle Sw|v \rangle = \left. \frac{d}{dt} \right|_{t=0} \langle U(t)v|w \rangle \frac{d}{dt} \left. \langle w|U(-t)v \rangle = -\langle w|Lv \rangle$$  

this implies that $w \in \widetilde{H}^\infty$ and $iL = iS = \overline{iL}$. Now if we apply the above argument we see that $\widetilde{H}^\infty = \widetilde{H}^\infty$. ■
This implies that we have via the spectral theorem

\[ W(t) = e^{-it(iL)} \]

a unitary representation of \( \mathbb{R} \). We assert that \( W(t) = U(t) \). Indeed, we have \( \mathcal{H}^\infty \subset \mathcal{H}^\infty \). So if \( v \in \mathcal{H}^\infty \) we have \( t \mapsto W(t)v \) and \( t \mapsto U(t)v \) are both solutions to the differential equation \( \frac{d}{dt}f(t) = Lf(t) \), \( f(0) = v \). Thus \( W(t) \) and \( U(t) \) agree on the dense subspace \( \mathcal{H}^\infty \) so they are equal. We have proved

**Theorem 4** Let \( (U, \mathcal{H}) \) be a unitary representation of \( \mathbb{R} \) then there exists a self adjoint operator, \( H \), on \( \mathcal{H} \) such that

\[ U(t) = e^{-iHt}. \]

Furthermore, if we define \( \mathcal{H}^\infty \) to be the space of \( v \in \mathcal{H} \) such that the map \( t \mapsto U(t)v \) is of class \( C^\infty \) and if \( Lv = \frac{d}{dt}U(t)v|_{t=0} \) for \( v \in \mathcal{H}^\infty \) then \( H \) is the closure of \( iL \) (which has domain \( \mathcal{H}^\infty \)) and \( \mathcal{H}^\infty \) is the space of all \( v \in \mathcal{H} \) such that \( t \mapsto U(t)v \) is of class \( C^1 \).