6 The solution to the quantum harmonic oscillator

We maintain the notation of the previous section. We are studying the quantum Harmonic oscillator

$$H = \frac{-\hbar^2}{2m}\frac{d^2}{dq^2} + \frac{kq^2}{2}$$

with $\omega = \sqrt{\frac{k}{m}}$. We have found an orthonormal basis of $L^2(\mathbb{R})$, ψ_n, $n = 0, 1, ...$ with $\psi_n \in \mathcal{S}(\mathbb{R})$ and

$$H\psi_n = \hbar\omega(n + \frac{1}{2})\psi_n.$$

Exercise. The domain of $\overline{H}$ is the space of all $f \in L^2(\mathbb{R})$ with

$$\sum |\langle \psi_n | f \rangle|^2 (\frac{1}{2} + n)^2 < \infty.$$

(Hint use the Riesz representation theorem.) Use this to see that the closure is self adjoint.

We can thus form the solution to the Schrödinger equation with initial condition $\phi(0)$

$$\phi(t) = e^{-\frac{i}{\hbar}tH}\phi(0) =$$
$$\sum_{n \geq 0} e^{-it\omega(\frac{1}{2}+n)} \langle \psi_n | \phi(0) \rangle \psi_n.$$

Notice that the phases that come into the quantum solution are essentially the same as those in the classical solution.

7 Unitary representations of $\mathbb{R}$

Let $\mathcal{H}$ be a Hilbert space and let $End(\mathcal{H})$ be the space of all bounded operators on $\mathcal{H}$. The strong operator topology on $End(\mathcal{H})$ is defined by the seminorms $\mu_v(A) = \|Av\|$ for all $v \in \mathcal{H}$.

If G is a topological group then a unitary representation of G on a Hilbert space $\mathcal{H}$ is a homomorphism, U, of G to $U(\mathcal{H})$ that is continuous in the strong operator topology. This is the same as saying that the map $G \times \mathcal{H} \to \mathcal{H}$ given by $g, v \mapsto U(g)v$ is continuous. To see this we note that $\|U(g)v - U(g_o)v_o\| = \|(U(g) - U(g_o))v_o + U(g)(v - v_0)\| \leq \mu_{v_o}(U(g) - U(g_o)) + \|v - v_o\|$.

Lemma 1 *Let $\mathcal{D}$ be a dense subspace of $\mathcal{H}$ such that the maps $G \to \mathcal{H}$ given by $g \mapsto U(g)v$ are continuous for $v \in \mathcal{D}$. Then $U : G \to U(\mathcal{H})$ is strongly continuous.*

Proof. Let $v_o \in \mathcal{D}$ and $v \in \mathcal{H}$ then

$$\mu_v(U(g) - U(g_o)) = \|U(g)v - U(g_o)v\| =$$

$$\|U(g)v_o - U(g_o)v_o + U(g)(v - v_o) + U(g_o)(v_o - v)\|$$
$$\leq \mu_{v_o}(U(g) - U(g_o)) + 2\|v - v_o\|.$$

■

In this section we will give a classification of unitary representations of $\mathbb{R}$. We first note that we have shown in section 3 that if H is a self adjoint operator on the Hilbert space $\mathcal{H}$ and if P is the corresponding spectral measure. Then if v is in the domain of $\overline{H}$

$$v \mapsto \int_{\mathbb{R}} e^{it\lambda} dP(\lambda)v$$

extends to a unitary representation of $\mathbb{R}$. We prove that the converse is also true.

Let $(U, \mathcal{H})$ be a unitary representation of $\mathbb{R}$. We define $\mathcal{H}^\infty$ to be the space of all $v \in \mathcal{H}$ such that the map $t \mapsto U(t)v$ defines a C^∞ map of $\mathbb{R}$ to $\mathcal{H}$. We note that $\mathcal{H}^\infty$ is $U(\mathbb{R})$ invariant. We define an operator $L : \mathcal{H}^\infty \to \mathcal{H}^\infty$ by $Lv = \frac{d}{dt}_{t=0} U(t)v$.On $\mathcal{H}^\infty$ we define the seminorms

$$\psi = \xi_k(v) = \left\|L^k v\right\|.$$

Lemma 2 *$\mathcal{H}^\infty$ endowed with these seminorms is a Fréchet space.*

Proof. We need only show that it is complete. Let $\{v_j\}$ be a Cauchy sequence in $\mathcal{H}^\infty$ then using ξ_0 and ξ_1 we see that there exists $w \in \mathcal{H}$ such that $\lim_{j\to\infty} v_j = w$ in $\mathcal{H}$ and $\lim_{j\to\infty} Lv_j = w_1$ in $\mathcal{H}$. Now

$$\frac{d}{dt} U(t)v_j = U(t)Lv_j.$$

This implies

$$U(t)v_j = \int_0^t U(s)Lv_j ds - v_j$$

taking the limit as $j \to \infty$ we have

$$U(t)w = \int_0^t U(s)w_1 ds - w.$$

This implies that $t \mapsto U(t)w$ is of class C^1 and $\frac{d}{dt}U(t)w = U(t)w_1$. We can now apply the same argument to w_1 using ξ_1 and ξ_2 to see that the map is C^2 and the second derivative is given by $\lim_{t\to\infty} L^2 v_j$. The result follows from the obvious inductive argument. ■

We note that if $v, w \in \mathcal{H}^\infty$ then

$$0 = \frac{d}{dt} \langle U(t)v|U(t)w\rangle_{|t=0} = \langle Lv|w\rangle + \langle v|Lw\rangle .$$

Thus iL is symmetric with domain $\mathcal{H}^\infty$.

Lemma 3 *$\overline{iL}$ is self adjoint.*

Proof. Let $\widetilde{\mathcal{H}^\infty}$ be the set of all $v \in \mathcal{H}$ such the map $w \mapsto \langle v|iLw\rangle$ on $\mathcal{H}^\infty$ extends to a continuous map of $\mathcal{H}$ to $\mathbb{C}$. We note that if $w \in \widetilde{\mathcal{H}^\infty}$ and if $v \in \mathcal{H}^\infty$ then

$$\left\langle \int_0^t U(s)\overline{L}w ds - w|v \right\rangle = \int_0^t \left\langle U(s)\overline{L}w|v\right\rangle ds - \langle w|v\rangle =$$

$$\int_0^t \left\langle \overline{L}w|U(s^{-1})v\right\rangle ds - \langle w|v\rangle = -\int_0^t \left\langle w|U(s^{-1})Lv\right\rangle ds - \langle w|v\rangle$$
$$= \left\langle w|U(t^{-1})v\right\rangle = \langle U(t)v|w\rangle .$$

Since $\mathcal{H}^\infty$ is dense we see that $\int_0^t U(s)\overline{L}w ds - w = U(t)w$ for $w \in \widetilde{\mathcal{H}^\infty}$. We therefore have shown that if $w \in \widetilde{\mathcal{H}^\infty}$ then $U(t)w$ is of class C^1. If $w \in \mathcal{H}$ is such that $t \mapsto U(t)w$ is of class C^1 then we define $Sw = \frac{d}{dt}U(t)w_{|t=0}$ then if $v \in \mathcal{H}^\infty$

$$\langle Sw|v\rangle = \frac{d}{dt}_{t=0} \langle U(t)w|v\rangle \frac{d}{dt}_{t=0} \langle w|U(-t)v\rangle = -\langle w|Lv\rangle$$

this implies that $w \in \widetilde{\mathcal{H}^\infty}$ and $\overline{iL} = iS = \overline{\overline{iL}}$. Now if we apply the above argument we see that $\widetilde{\widetilde{\mathcal{H}^\infty}} = \widetilde{\mathcal{H}^\infty}$. ■

This implies that we have via the spectral theorem

$$W(t) = e^{-it(\overline{iL})}$$

a unitary representation of $\mathbb{R}$. We assert that $W(t) = U(t)$. Indeed, we have $\mathcal{H}^\infty \subset \widetilde{\mathcal{H}^\infty}$. So if $v \in \mathcal{H}^\infty$ we have $t \mapsto W(t)v$ and $t \mapsto U(t)v$ are both solutions to the differential equation $\frac{d}{dt}f(t) = Lf(t)$, $f(0) = v$. Thus $W(t)$ and $U(t)$ agree on the dense subspace $\mathcal{H}^\infty$ so they are equal. We have proved

Theorem 4 *Let $(U, \mathcal{H})$ be a unitary representation of $\mathbb{R}$ then there exists a self adjoint operator, H, on $\mathcal{H}$ such that*

$$U(t) = e^{-iHt}.$$

Furthermore, if we define $\mathcal{H}^\infty$ to be the space of $v \in \mathcal{H}$ such that the map $t \mapsto U(t)v$ is of class C^∞ and if $Lv = \frac{d}{dt}U(t)v_{|t=0}$ for $v \in \mathcal{H}^\infty$ then H is the closure of iL (which has domain $\mathcal{H}^\infty$) and $\widetilde{\mathcal{H}^\infty}$ is the space of all $v \in \mathcal{H}$ such that $t \mapsto U(t)v$ is of class C^1.