3 Jan 2011

"Groups," lie looks at.

\[ 0 \in U \subseteq \mathbb{R}^n \]

\[ m: U \times U \rightarrow \mathbb{R}^n \text{ \ open } \int \text{ real analytic} \]

\[ \eta: U \rightarrow U \]

\[ m(x, 0) = m(0, x) = x \quad \forall x \in U \]

\[ m(\eta(x), x) = m(x, \eta(x)) = 0 \]

\[ m(x, m(y, z)) = m(m(x, y), z) \text{ if both sets make sense} \]

and it is called local analytic group.

\section*{Lie algebra}

Consider \[ X: U \rightarrow \mathbb{R}^n \text{ real analytic } \quad 0 \in W \subseteq U \]

\[ d m(x, \cdot) y X(y) = X(m(x, y)) \]

Will prove: \[ \text{lie}(G) = \text{the set of all these } \text{"functions".} \]

\[ \dim_{\mathbb{R}} \text{lie}(G) = n \]

If \[ X = (x_1, \ldots, x_n), \ Y = (y_1, \ldots, y_n), \]

\[ [X, Y]_j = \sum_{k=1}^n x_k \frac{\partial y_j}{\partial x_k} - \sum_{k=1}^n y_k \frac{\partial x_j}{\partial x_k} \]

\[ [X, Y] \in \text{lie}(G) \]

If \( g \) is an \( n \) dim vector space over \( \mathbb{R} \) with \([,]\) bilinear op. satisfy (1), (2), then \( g \) is called a Lie algebra over \( \mathbb{R} \)

\[ \begin{align*}
(1) \quad [X, Y] = -[Y, X] & \iff [X, X] = 0 \quad (\text{for } \forall x \in g) \\
(2) \quad [X, [Y, Z]] & = [[X, Y], Z] + [Y, [X, Z]] \quad \text{Jacobi-identity}
\end{align*} \]
Lie's third thm
If \( g \) is an \( n \)-dim lie alg. over \( \mathbb{R} \),
then \( g \) is isomorphic with \( \text{lie}(G) \) for \( G \) local analytic gp

Lie's second thm
If \( L: \text{lie}(G) \rightarrow \text{lie}(H) \) is a lie alg homo.
then there "exists" \( \Psi: G \rightarrow H \) local lie gp. homo
such that for \( x \in \text{lie}(G) \),
\[
d\Psi_x(X(x)) = L(X)(\Psi(x))
\]

Transformation groups
\( G \) (n dim) acts on \( \mathbb{R}^m \)
\[
F: V \times \mathbb{Z} \rightarrow \mathbb{R}^m \text{ real analytic, } \quad \text{open}
\]
satisfy
1. \( F(0,z) = z \), \( z \in V \)
2. \( F(x,F(1,y),z)) = F(m(x,y),z) \)
\[
( F(\eta(x),F(x,z)) = z )
\]

Autonomous System of 1st order equation
\[
(*) \quad \frac{dx}{dt} = X(x), \quad X : U \rightarrow \mathbb{R}^n \text{ analytic}
\]

\((x, \ldots, x)\) : \text{analytic in } t.

Lie's question: How to solve it in simplest way? (not existence problem)

Define a gp. of symmetries of the eqn
lie gp. \( G \) : transformation gp of \( \mathbb{R}^n \) s.t. \( X(F(z,x)) = X(x) \)

Thus: \( (*) \) can be solved by quadrature if \( G \) exists and is solvable.

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Textbook: Symmetry, Representations and Invariants

Prerequisites:
1. Basic point set topology
2. Advanced Calculus (Inverse fun thm, existence & uniqueness of ODE)
3. Manifold theory (Partition of unity, vector field, integration)