Lie Group

Note Title 1/3/2011

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"Groups" Lie looks at.

$0 \in U \subseteq \mathbb{R}^n$

$m: U \times U \to \mathbb{R}^n$ U open $\Big\}$ real analytic

$\eta: U \to U$

$m(x,0) = m(0,x) = x \quad \forall x \in U$

$m(\eta(x), x) = m(x, \eta(x)) = 0$

$m(x, m(y,z)) = m(m(x,y), z)$ if both sets make sense

and it is called local analytic group.

Lie algebra

Consider $X: U \to \mathbb{R}^n$ real analytic $0 \in W \overset{\text{open}}{\subseteq} U$

$$dm(x,\cdot)_y \, X(y) = X(m(x,y))$$

Will prove: $\mathrm{Lie}(G)$ = the set of all these "functions".

$$\dim_{\mathbb{R}} \mathrm{Lie}(G) = n$$

If $X = (X_1, \ldots, X_n)$, $Y = (Y_1, \ldots, Y_n)$,

$$[X,Y]_j = \sum_{i=1}^n X_i \frac{\partial Y_j}{\partial x_i} - \sum_{i=1}^n Y_i \frac{\partial X_j}{\partial x_i}$$

$$[X,Y] \in \mathrm{Lie}(G)$$

If $\mathfrak{g}$ is an n dim vector space over $\mathbb{R}$ with $[\,,\,]$ bilinear op. satisfy (1), (2), then $\mathfrak{g}$ is called a Lie alg over $\mathbb{R}$

(1) $[X,Y] = -[Y,X] \Leftrightarrow [X,X] = 0$ (for char $\neq 2$)

(2) $[X,[Y,Z]] = [[X,Y],Z] + [Y,[X,Z]]$ Jacobi-identity

Lie's third thm

If $\mathfrak{g}$ is an n-dim Lie alg. over $\mathbb{R}$.

then $\mathfrak{g}$ is isomorphic with $\mathrm{Lie}(G)$ for G local analytic gp

Lie's second thm

If $L: \mathrm{Lie}(G) \to \mathrm{Lie}(H)$ is a Lie alg homo.

then there "exists" $\varphi: G \to H$ local Lie gp. homo

s.t. for $X \in \mathrm{Lie}(G)$, $d\varphi_x(X(x)) = L(X)(\varphi(x))$

Transformation groups ($\sim$ permutation gp in Galois theory)

G (n dim) acts on $\mathbb{R}^m$

$F: V \times Z \longrightarrow \mathbb{R}^m$ real analytic, $0 \in V \overset{\text{open}}{\subseteq} U$, $Z \subseteq \mathbb{R}^m$ is open

Satisfy 1. $F(0, z) = z$, $z \in V$

2. $F(x, F(y, z)) = F(m(x, y), z)$

$(F(\eta(x), F(x, z)) = z)$

Autonomous System of 1st order equation

$(*)$ $\frac{dx}{dt} = X(x)$, $X: U \longrightarrow \mathbb{R}^n$ analytic

$(x_1, \dots, x_n)$ x_i analytic in t.

Lie's question: How to solve it in simplest way? (not existence problem)

Define a gp. of symmetries of the eqn

Lie gp. G: transformation gp of $\mathbb{R}^n$ s.t. $X(F(z, x)) = X(x)$

Thm: $(*)$ can be solved by quadrature iff G exists and is solvable.

Textbook: Symmetry, Representations and Invariants

Pre-requisites: 1. Basic point set topology

2. Advanced Calculus (Inverse fcn thm, existence & uniqueness of ODE)

3. Manifold theory (Partition of unity, vector field, Integration)