Thm  

G locally compact topological group

Let \( B \) be the \( \sigma \)-algebra generated by the open subsets

Then on \( B \) there is a unique countably additive left invariant, positive measure up to positive scalar.

If \( g \in G \), \( \mu \) is a Haar measure on \( G \), locally compact group

\[
R_g^* \mu = \mu \circ R_g^* \quad \quad R_g^* f = f \circ R_g
\]

Therefore implies \( R_g^* \mu = c(g) \mu \).

\( c: G \to \mathbb{R}_{>0} \) is constant.

\( c(g) = S(g)^{-1} \), where \( S \) is modular function of \( G \).

\( G \) Lie group, \( X \in \text{Lie}(G) \)

\[
dR_g X = Y(g) g^{-1} \quad Y(g) \in \text{Lie}(G)
\]

\[
d\text{log}_g \circ dR_g X = Y(g) e \quad (d\text{log}_g \circ dR_g) X = Y(g)
\]

\[
\text{Int}(g) X = g X g^{-1}
\]

\[
d\text{Int}(g) X = d\text{Int}(g) = \text{Ad}(g)
\]

\[
W_g(X_1, \ldots, X_n) = W(X_1, \ldots, X_n) \quad \text{we } \mathbb{N}^* \text{Lie}(G)^* \{0\}
\]

\[
R_g^* w = \det(\text{Ad}(g))^{-1} w
\]

Corresponding measure \( \mu \) is such that \( R_g^* \mu = \det(\text{Ad}(g))^{-1} \mu \)

\[
\Rightarrow S(g) = \left| \det(\text{Ad}(g)) \right|
\]

\( G \) is said to be unimodular if \( S(g) = 1 \).

Examples: Observe \( S: G \to \mathbb{R}_{>0} \) is a continuous gp. homomorphism

1. If \( [G,G] = G \), then \( S = 1 \)

2. If \( G \) is abelian, \( S = 1 \) \( \quad \therefore R_g = L_g \)

3. If \( G \) is compact, \( S = 1 \), \( S: G \to \mathbb{R}_{>0} \) is cts gp. homo \( \therefore S(R_o) \) is cts
\[ G = \left\{ \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \mid a > 0, \ b \in \mathbb{R} \right\} \]

\[ L^2(G, \mu) \quad \langle f_1, f_2 \rangle = \mu(f_2^* f_1) \quad f_i \in C(G; C) \]

A unitary representation of a locally compact topological group is a pair \((\pi, H)\) where \(H\) is a Hilbert space and \(\pi: G \to U(H) = \{ A: H \to H \mid A \text{ is a unitary operator} \}\)

\[ \langle \pi(v)A_w, \pi(v)w \rangle = \langle v, w \rangle + v, w, \text{ A bijective} \]

\[ G \times H \to H \text{ given by} \]
\[ (g, v) \mapsto \pi(g)v \text{ is continuous} \]

\[ \text{(cts in strong top.)} \]

\[ \iff v \in H, \ G \to H \text{ is continuous} \]
\[ g \mapsto \pi(g)v \]

E.g. If \(H\) is a finite dimensional Hilbert space and \(\pi: G \to U(H)\) is \(g\) hom\(\overline{\text{om}}\)

is \(\text{cts in usual sense then it is a unitary rep and conversely.} \)

Example: \(H = L^2(G, \mu)\) fixed left inv. positive measure

\[ \int_G f(g)dg = \mu_c(f) \]

\[ L^2(G), \text{ define } \pi(g) = L^*_g, \]
\[ \pi(g_1, g_2) = \pi(g_1)\pi(g_2) \]
\[ \langle \pi(g)f_1, \pi(g)f_2 \rangle = \int_G f_1(g^* x)\overline{f_2(g^* x)}dx \]
\[ \mu_c(L^*_g f, f) \]
\[ = \langle f_1, f_2 \rangle \]
A unitary representation \( (\pi, L^2(G)) \) is a unitary gp representation follows from the fact that a continuous functions on a compact set is uniformly continuous.

A unitary representation \( (\pi, H) \) of \( G \) is said to be irreducible if whenever \( \mathcal{V} \subseteq H \) is closed and \( \pi(g) \mathcal{V} \subseteq \mathcal{V} \; \forall g \in G \), then \( \mathcal{V} = \text{span} \mathcal{V} \) or \( \mathcal{V} = H \).

Thm If \( G \) is compact and \( \mathcal{V} \subseteq L^2(G) \) is closed and the action of \( G \) on \( \mathcal{V} \) is irreducible, then \( \dim \mathcal{V} < \infty \).

Pf: Let \( f \in \mathcal{V} \), \( f \) a unit vector.

\[ P_f = \langle v, f \rangle f \]

Define \( \mathcal{V} (v, w) = \int_G \langle \pi(g) P \pi(g)^{-1} v, w \rangle dg \)

\( \mathcal{V} \) is sesquilinear on \( \mathcal{V} \times \mathcal{V} \)

I.e. linear in first variable, conjugate linear in second.

\[ |\mathcal{V}(v, w)| \leq C \|v\| \|w\| \]

\[ (\therefore \int_G \langle \pi(g) P \pi(g)^{-1} v, w \rangle \leq \|P\| \|v\| \|w\| \mu(1_f) \geq \mu(f) \]}

\[ \Rightarrow \exists Q: H \rightarrow H \text{ bounded s.t. } \langle Qv, w \rangle = \mathcal{V}(v, w) \]

\[ Q \pi(g) = \pi(g) Q \]

\[ \langle \pi(g)^{-1} Q \pi(g) v, w \rangle = \langle Q \pi(g) v, \pi(g) w \rangle \]

\[ = \int_G \langle \pi(g) P \pi(g)^{-1} \pi(g) v, \pi(g) w \rangle dg \]

\[ = \int_G \langle \pi(g) P \pi(g)^{-1} v, w \rangle dg \text{ left invariant} \]

\[ = \langle Qv, w \rangle \]

1) \( Q \neq 0 \), \( Q^* = Q \) self-adjoint

2) \( Q \) is a compact operator
Q: $H \rightarrow H$ is compact means that if $B \subseteq H$ is bounded, then $QB$ is compact.

**Spectral theorem for compact self-adjoint operator:**

**H** have countable orthogonal basis

$T: H \rightarrow H$ self-adjoint

Then $H = H_0 \oplus \left( \bigoplus_{i=1}^{N} H_i \right)$ $0 \leq N < \infty$

$\langle H_i, H_j \rangle = 0$ if $i \neq j$

and $T \mid H_j = \lambda_i I$, $\lambda_i > 0$ $\lambda_0 = 0$

$\lambda_i \in \mathbb{R}$ and $\dim H_j < \infty$ for $j \geq 1$.

$\Rightarrow \lambda_i \rightarrow 0$

$\quad \quad \quad \Rightarrow V_i \ni 0$

$v = V \lambda_i$, $Q \pi(g) v = \pi(g) Q v = \lambda_i \pi(g) v$

$\Rightarrow \pi(g) v \lambda_i \in V \lambda_i$, $g \in G$

$\Rightarrow V \lambda_i \ni 0$, $V \lambda_i = V_i$

$\Rightarrow \lambda_i I$ is compact on $V$

$\Rightarrow V$ is finite dim.