Practice Final Examination Mathematics 100A

1. Let $G = \left\{ g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} | a,b,c,d \in \mathbb{Z}_3 \text{ and } ad - bc = \overline{1} \right\}$.

(a) Prove that G has order 24. Hint: Set $H = \left\{ g = \begin{bmatrix} \overline{1} & x \\ 0 & \overline{1} \end{bmatrix} | x \in \mathbb{Z}_3 \right\}$. Show that $\phi : G/H \to \{(x,y) | (x,y) \neq (\overline{0},\overline{0})\}$ given by $\phi(gH) = (a,c)$ for $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is well defined and bijective. (Hint: Show that if $m = \begin{bmatrix} a & b' \\ c & d' \end{bmatrix}$ and $ad' - b'c = 1$ then $g^{-1}m \in H$ so $m \in gH$.)

(b) Prove that G is not isomorphic with S_4.

2. Let $n \geq 4$ consider the group H generated by all 4-cycles in S_n. Show that $H = S_n$. Hint: First show that H is a normal subgroup. Then show that $(1432)(1423)(1243) = (12)$.

3. Which of the following sets with binary operations as given (the a operation is for the addition the m for multiplication) is a ring with the indicated 0_R and 1_R? (You must give reasons to get full credit.)

a) $R = \mathbb{Z}$ and $a(x,y) = x - y$ and $m(x,y) = xy$ with $0_R = 0$ and $1_R = 1$.

b) $R = \mathbb{Z}$ and $a(x,y) = x + y + 1$ and $m(x,y) = xy + x + y$ with $0_R = -1$ and $1_R = 0$.

c) $R = \mathbb{R} \times \mathbb{R}$ and $a((x_1,x_2),(y_1,y_2)) = (x_1+y_1, x_2+y_2)$, $m((x_1,x_2),(y_1,y_2)) = (x_1y_1 + x_2y_2, x_1y_2 + x_2y_1)$ with $0_R = (0,0), 1_R = (1,0)$.

4. Let G be a commutative group and let H denote the set of elements of finite order in G.

a) Show that H is a subgroup of G.

b) Consider the matrices $A = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ calculate the orders of A and B and deduce that the elements are of finite order. Show that AB does not have finite order. Why doesn't this contradict part a)?

5. Assume that G is a group whose order is 10 show that G is isomorphic to ether $\mathbb{Z}_{10}$ or D_5 (the dihedral group with 10 elements).

6. Let G be a group and let H be a cyclic subgroup of G that is normal in G show that every subgroup of H is normal in G.

7. Let R be a commutative ring and S be the set of 2×2 matrices with entries in R under matrix multiplication. Show that S is a ring. Show that

$$S^* = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} | a,b,c,d \in R \text{ and } ad - bc \in R^* \right\}.$$

8. Let G be a finite group such that if $g \in G$ then $g^2 = e$ (the identity element).

a) Prove that G is abelian.

b) Prove that $|G| = 2^n$ for some n.

c) Prove that if $|G| = 2^n$ then G is isomorphic with the product group $C_2 \times C_2 \times \cdots \times C_2$ (n-copies). Hint: Prove by induction on n that if $|G| = 2^n$ then there exists an onto group homomorphism from G to C_2. (To do this observe that if $x \in G$ and $x \neq e$ then $H = \{e, x\}$ is a normal subgroup and G/H has order 2^{n-1}. If there exists an onto homomorphism, ϕ, of G/H ont C_2 then $\eta(g) = \phi(gH)$ is an onto homomorphism of G to C_2.) Write $C_2 = \{e, a\}$. Let η be an onto homomorphisms of G to C_2 and let U be the kernel of η. If $x \in G$ is such that $\eta(x) = a$ then show that the map $\alpha : C_2 \times U \to G$ given by $\alpha(e, u) = u$ and $\alpha(a, u) = xu$ defines a group homomorphism.