

Solutions Math 100A Practice Midterm Fall 2003

1. a) This relation is symmetric and reflexive but not transitive. To show that it is not transitive notice that $a \equiv 0$ for all a . Thus $2 \equiv 0$ and $3 \equiv 0$ so $0 \cdot 3 = 0$ hence $0 \equiv 3$. But $2 \not\equiv 3$ and $2 \cdot 3 \neq 0$.

b) Reflexive: $a \equiv a$ hence $a \equiv a$.

Symmetric: $a \equiv b$ implies that $a = b$ or $ab = 1$. This implies that $b = a$ or $ba = 1$.

Transitive: $a \equiv b$ and $b \equiv c$ implies that $a = b$ or $ab = 1$ and $b = c$ or $bc = 1$. If $a = b$ and $b = c$ then $a = c$ so $a \equiv c$ in this case. If $a = b$ and $bc = 1$ then $ac = bc = 1$ so $a \equiv c$ in this case. If $a \neq b$ then $ab = 1$. If $b = c$ then $ac = 1$ so $a \equiv c$. In the last case $a \neq b$ and $b \neq c$ thus $ab = 1$ and $bc = 1$. Thus $b = a^{-1}$ and $c = b^{-1}$ hence $c = a$. Thus $a \equiv c$. Thus the relation is an equivalence relation.

c) Reflexive: $s^2 = t^2$.

Symmetric: If $s^2 = t^2$ then $t^2 = s^2$.

Transitive: If $s^2 = t^2$ and $t^2 = u^2$ then $s^2 = u^2$.

The equivalence class of t consists of all s with $s^2 = t^2$. If $t \neq 0$ then the equation $x^2 = t^2$ has two solutions $x = t$ and $x = -t$ if $t = 0$ then the only solution is $x = 0$. Thus the set of equivalence classes is $\{\{0\}\} \cup \{\{t, -t\} | t > 0\}$.

2. a) Suppose that $a \in \mathbb{Z}$ and $a + x + 2 = x$ for all $x \in \mathbb{Z}$. Then solving for a we have $a = -2$. Thus there is an identity, $e = -2$. If $a, b, c \in \mathbb{Z}$ then

$$a * (b * c) = a + b * c + 2 = a + (b + c + 2) + 2 = a + b + c + 4.$$

Also

$$(a * b) * c = (a * b) + c + 2 = (a + b + 2) + c + 2 = a + b + c + 4.$$

Thus this binary operation is associative. Suppose $a \in \mathbb{Z}$ then we attempt to solve $a * x = -2$. That is $a + x + 2 = -2$. Then $x = -a - 4$. Since $a + (-a - 4) + 2 = -2$ we can take the inverse of a to be $-a - 4$. We have proved that this example is a group.

b) Suppose that $a = x * a$ for all $a \in \mathbb{Z}$. Then $xa + 2x + 2a + 2 = a$ for all a . hence

$$x(a + 2) = -(a + 2).$$

One solution is $x = -1$. Thus we can take $e = -1$. As for the associative rule we have

$$\begin{aligned} a * (b * c) &= a(b * c) + 2a + 2b * c + 2 = \\ &= a(bc + 2b + 2c + 2) + 2a + 2(bc + 2b + 2c + 2) + 2 = \\ &= abc + 2ab + 2ac + 2a + 2a + 2bc + 4b + 4c + 4 + 2 = \\ &= abc + 2ab + 2ac + 2bc + 4a + 4b + 4c + 6. \end{aligned}$$

We now calculate

$$\begin{aligned} (a * b) * c &= (ab + 2a + 2b + 2) * c = \\ &= (ab + 2a + 2b + 2)c + 2(ab + 2a + 2b + 2) + 2c + 2 = \\ &= abc + 2ac + 2bc + 2c + 2ab + 4a + 4b + 4 + 2c + 2 = \\ &= abc + 2ac + 2bc + 2ab + 4a + 4b + 4c + 6. \end{aligned}$$

The two expressions are equal so the associative rule is true. Now suppose that $a \in \mathbb{Z}$ then an inverse of a must satisfy $x * a = -1$. That is $xa + 2a + 2x = -1$. That is

$$x(a + 2) = -2a - 1.$$

If $a = -2$ then this says $x \cdot 0 = 3$. We can't solve this equation so in this case we only have a monoid.

c) An identity element must satisfy $x * a = a$ for all a . That is $x^2 a^2 = a$ for all a . If $a \neq 0$ then this says that $x^2 a = 1$ for all a . There is no such x since if $a = -1$ then there would have to be an $x \in \mathbb{Z}$ with $x^2 = -1$. Relative to this binary operation \mathbb{Z} is not a monoid.

3. In \mathbb{Z}_{17} , $\overline{16} = \overline{(-1)}$. We observe that if $a \in \mathbb{Z}$ then in \mathbb{Z}_n $\overline{a}^n = \overline{a^n}$. We prove this by induction on n . If $n = 1$ then $\overline{a}^1 = \overline{a} = \overline{a^1}$. Assume for $n = m$. Then $\overline{a}^{m+1} = \overline{a}^m \overline{a}^1 = \overline{a^m} \cdot \overline{a} = \overline{a^m a} = \overline{a^{m+1}}$. The assertion is thus true for all n . In the case at hand this says that $\overline{16}^{27} = \overline{(-1)^{27}} = \overline{(-1)}$.

4. $\mathbb{Z}_8 = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}, \overline{6}, \overline{7}\}$. It's Cayley table under multiplication is

	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$	$\overline{5}$	$\overline{6}$	$\overline{7}$
$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$
$\overline{1}$	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$	$\overline{5}$	$\overline{6}$	$\overline{7}$
$\overline{2}$	$\overline{0}$	$\overline{2}$	$\overline{4}$	$\overline{6}$	$\overline{0}$	$\overline{2}$	$\overline{4}$	$\overline{6}$
$\overline{3}$	$\overline{0}$	$\overline{3}$	$\overline{6}$	$\overline{1}$	$\overline{4}$	$\overline{7}$	$\overline{2}$	$\overline{5}$
$\overline{4}$	$\overline{0}$	$\overline{4}$	$\overline{0}$	$\overline{4}$	$\overline{0}$	$\overline{4}$	$\overline{0}$	$\overline{4}$
$\overline{5}$	$\overline{0}$	$\overline{5}$	$\overline{2}$	$\overline{7}$	$\overline{4}$	$\overline{1}$	$\overline{6}$	$\overline{3}$
$\overline{6}$	$\overline{0}$	$\overline{6}$	$\overline{4}$	$\overline{2}$	$\overline{0}$	$\overline{6}$	$\overline{4}$	$\overline{2}$
$\overline{7}$	$\overline{0}$	$\overline{7}$	$\overline{6}$	$\overline{5}$	$\overline{4}$	$\overline{3}$	$\overline{2}$	$\overline{1}$

We note that the rows (or columns) that contain $\overline{1}$ are labeled by $\overline{1}, \overline{3}, \overline{5}, \overline{7}$. These are therefore the invertible elements. By deleting the row and column for the other elements we get the Cayley table for \mathbb{Z}_8^* .

	$\overline{1}$	$\overline{3}$	$\overline{5}$	$\overline{7}$
$\overline{1}$	$\overline{1}$	$\overline{3}$	$\overline{5}$	$\overline{7}$
$\overline{3}$	$\overline{3}$	$\overline{1}$	$\overline{7}$	$\overline{5}$
$\overline{5}$	$\overline{5}$	$\overline{7}$	$\overline{1}$	$\overline{3}$
$\overline{7}$	$\overline{7}$	$\overline{5}$	$\overline{3}$	$\overline{1}$

Consider $a = \overline{3}, b = \overline{5}$ then $a^0 b^0 = \overline{1} \cdot \overline{1} = \overline{1}, a^1 b^0 = a \cdot \overline{1} = a = \overline{3}, a^0 b^1 = b = \overline{5}, a^1 b^1 = \overline{3} \cdot \overline{5} = \overline{7}$. This gives every element. Any of the three pairs $\{\overline{3}, \overline{5}\}, \{\overline{3}, \overline{7}\}, \{\overline{5}, \overline{7}\}$ would have worked.

5. If $ab = ba$ and $a^2 = b^2 = e$ (the identity element) then applying the associative rule we have

$$(ab)^2 = (ab)(ab) = ((ab)a)b.$$

Applying it again we have

$$((ab)a)b = (a(ba))b.$$

we now use the fact that $ba = ab$ so

$$(a(ba))b = (a(ab))b.$$

Apply the associative rule again and have $((aa)b)b = bb = e$.

The following argument would have gotten full credit for this part since we already know that it doesn't matter how we place the parentheses in the multiplication. $(ab)^2 = abab = a(ba)b = a(ab)b = aabb = e$.

$$g = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, h = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}. \text{ Then } g^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \text{ and}$$

$$h^2 = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1-1 & -1 \cdot -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \text{ Now}$$

$$gh = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\text{so } (gh)^2 = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix},$$

$$(gh)^3 = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},$$

$$(gh)^4 = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix},$$

$$(gh)^5 = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix},$$

$$(gh)^6 = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

This is not a contradiction to the first part of the problem since $gh \neq hg$.