Try: If \( f : S^2 \to \mathbb{R}_{\geq 0} \) is a frame function, then \( f \) is continuous.

If \( \psi : S^2 \to \mathbb{R} \) is a bounded function, then “define” \( \text{Var}_T(\psi) = \text{sup}(\psi|_T) - \text{inf}(\psi|_T) \)

Main concept is of \( \text{EW} \) great circle.

\( S = S^2 \subset \mathbb{R}^3 \), \( p \) is non-pole.

\[ N = \{ x \mid x_{320} \} \]

\( \text{XEN-SP} \), \( \text{EW}(x) \) is the great circle joining \( x \) and \( x \times p \) \( (e_E = \text{equator}) \)

Ex: Show this is the same as Gleason's \( \text{EW} \) great circle parametrically it is given by

\[ t \mapsto \cos t x + \sin t \frac{x \times p}{\| x \times p \|} \]

pole of great circle \( \frac{x - (x \times p) x}{\| x - (x \times p) x \|} \)

\( \text{EW} \) through \( x \)

Lemma: If \( \text{ZEN-SP} \), then the set of all \( \text{XEN-SP} \)

s.t. there exists \( y \in \text{EW}(x) \cap \text{N-SP} \) s.t. \( z \in \text{EW}(y) \) has interior.

Hint: \( z \cdot (p - (y \times p) y) = 0 \)

and \( y \cdot (p - (x \times p) x) = 0 \) are the equations for the set of \((x,y)\)
\[ f>0 \text{ frame function} \]
\[ \text{Inf}(f) = u, \ u \ge 0 \]
may assume \( \text{Inf}(f) = 0 \) by subtracting \( u \).

Let \( \eta > 0 \) be given.
Let \( p \in S \) be such that \( f(p) < \eta \), make it the north pole.

\[ \sigma = \begin{pmatrix} \cos \frac{\pi}{2} & -\sin \frac{\pi}{2} & 0 \\ \sin \frac{\pi}{2} & \cos \frac{\pi}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \]

\[ x \in E, \quad \langle \sigma(x), x \rangle = 0 \quad (E \text{ the equator}) \]

\[ \Rightarrow \quad p, x, \sigma(x) \text{ is a frame (orthonormal basis)} \]

\[ f(x) + f(\sigma(x)) = w - f(p) \]

Define \( g(w) = f(w) + flow \), \( w \in S \)

\[ g|_E = w - f(p) \]

\( g \) is a frame function and weight is \( 2w \). constant on \( E \).

\[ r \in N - \frac{3}{2}p \]

\[ E \wedge (r) \cap E = F \text{ perpendicular to } r \]

\[ \Rightarrow \quad g(r) + g(q) \le 2w, \quad \text{since } q \in E \quad g(q) = w - f(p) \]

\[ \Rightarrow \quad g(r) + w - f(p) \le 2w \]

\[ \Rightarrow \quad g(r) \le w + f(p) \le w + \eta. \quad \text{Since } r \text{ is arbitrary in } N - \frac{3}{2}p \]

\[ \Rightarrow \quad \text{if } x \in N - \frac{3}{2}p, \text{ then } g(x) \le w + \eta \]
Note \( Ew(r) = SN \left( P \cap (r, r') \right) \)

\[ r, q \in Ew(r) \quad g(r) + g(q) = 2w - g \left( \frac{g'}{\|g'\|} \right) \]

Let \( s \in Ew(r) \), \( t \perp s \), \( t \in Ew(r) \)

Take \( t \in N - \{ p \} \)

Then \( g(t) = g(s) + g(t) \leq g(s) + w + \eta \)

So \( g(r) + w - f(p) = g(s) + g(t) \leq g(s) + w + \eta \)

\[ \Rightarrow g(r) \leq g(s) + 2\eta \quad \forall s \in Ew(r) \]

Take \( z \in N - \{ p \} \)

Let \( U \) be the open subset of \( N - \{ p \} \) from the lemma.

\[ x \in U, \exists y \in Ew(x) \text{ s.t. } z \in Ew(y) \]

If \( s \in Ew(r) \), then \( g(r) \leq g(s) + 2\eta \)

Let \( z \in N - \{ p \} \) such that

\[ g(z) < \inf_{z \in N - \{ p \}} g + \eta \]

\[ \beta \leq g(x) = g(y) + 2\eta \leq g(z) + 4\eta \leq \beta + 5\eta \]

So \( \forall x, \forall z \leq g(z) \leq 5\eta \)

**Lemma** 4: Frame function.

Let \( p \in S \) be such that \( p \) has a nbd \( T \) so that

\[ \forall x \in T, Var_x(U) = v \]

then if \( x \) is great circle with pole \( p \),

then \( \exists L \) nbd of \( x \) such that \( \forall x \in T \), \( Var_x(U) \leq 2v \)
\[
g(r) + g(r') = g(q_1) + g(q_1') \\
g(r) + g(r') = g(q_2) + g(q_2')
\]

\[
\Rightarrow g(q_1) - g(q_2) = g(r') - g(q_1') + g(q_2) - g(r') \\
\Rightarrow |g(q_1) - g(q_2)| \leq |g(r') - g(q_1')| + |g(q_2) - g(r')| \leq 2\eta \text{ since all the mined elements are above latitude } \theta.
\]

Thus there is a point \( q \) such that \( \text{Var}_L(g) \leq 2\eta \) is an open neighborhood.

Under some hypothesis, if \( r \in S \), then for some \( g \) such that \( \text{Var}_L(g) \leq 4\eta \):

\[
\text{Var}(g) \leq 4\eta \quad \text{if } r \in N \cap \{p\} \\
\text{otherwize } \quad r \text{ in } N \setminus \{p\}.
\]

\[
\exists R \text{ open with } p \in R \text{ with } \text{Var}_R(g) \leq 20\eta
\]

Know \( \text{Var}(p) = 2f(p) \)

\[
\therefore \text{Var}_R(p) \leq 22\eta
\]

If \( r \in S \), \( \exists \mod \text{ wrt } \theta \text{ such that } \text{Var}_{\theta}(f) \leq 88\eta \)
This implies the continuity so Gleason's theorem is (finally) proved.

There was one loose end. We asserted that if \( H^k \) is the space of spherical harmonics of degree \( k \) and if \( V \subset C(S^n) \) is a closed \( O(n+1) \)-invariant subspace then if \( \langle \psi | H^k \rangle \neq 0 \)
\( q^{-k} | V \rangle \). To see this we use the fact that if \( X_k(g) = tr(g | H^k) \) \( g \in O(n+1) \)
then \( dK = d\mu \circ H^k \). Then \( P_k : L^2(S^n) \to H^k | S^n \)
is given by \( P_k f(x) = dK \int X_k(g) f(g' x) dx \).

Let \( \overline{V} \) be the closure of \( V \) in \( L^2(S^n) \).

Then \( \langle \overline{V} | H^k | S^n \rangle \neq 0 \) so the Riesz representation theorem implies that \( \overline{V} \cap H^k | S^n \neq 0 \). Since \( O(n+1) \)
acts irreducibly \( H^k | S^n \subset \overline{V} \). Now we say that \( y, f \in H^k | S^n \) \( f, g \in V \) such that \( y \to f \) in \( L^2(S^n) \). From the formula for \( P_k \) above we see that \( P_k f \) is a
uniform limit of elements $f$ in $V$ for each $j$. Since $P_{k_1 f} \to f$ in $L^2(S^n)$ we see that $P_{k_1 f} \in V$ and $P_{k_1 f} \not\to f$ for $j$ sufficiently large. Thus $V \cap H^{1/2}_{S^n} \neq \emptyset$. So reducibility implies $H^{1/2}_{S^n} \subset V$. 