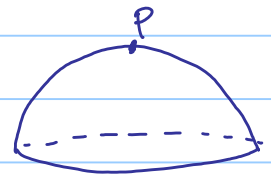
13 Nov, 2013

Thm If $f: S^2 \to \mathbb{R}_{\geq 0}$ is a frame function, then f is continuous.

If $\varphi: S^2 \to \mathbb{R}$ is a bounded function,
then "define" $\mathrm{Var}_T(\varphi) = \sup(\varphi|_T) - \inf(\varphi|_T)$
variation

Main concept is of EW great circle.

$S = S^2 \subset \mathbb{R}^3$, p is north pole.



$N = \{x \mid x_3 \geq 0\}$

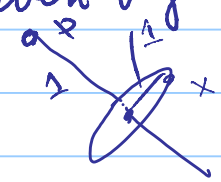
$x \in N - \{p\}$, $EW(x)$ is the great circle joining x and $\frac{x \times p}{\|x \times p\|}$ ($\in E$ = equator)

Ex Show this is the same as Gleason's EW great circle
parametrically it is given by

$t \mapsto \cos t\, x + \sin x\, t \frac{x \times p}{\|x \times p\|}$

pole of great circle $\frac{p - \langle x|p\rangle x}{\|p - \langle x|p\rangle x\|}$

EW through x



Lemma If $z \in N - \{p\}$,
then the set of all $x \in N - \{p\}$
s.t. there exists $y \in EW(x) \cap (N - \{p\})$ s.t. $z \in EW(y)$ has interior.

Hint: $z \cdot (p - \langle y|p\rangle y) = 0$
and $y \cdot (p - \langle x|p\rangle x) = 0$ are the equations for the set of (x, y)

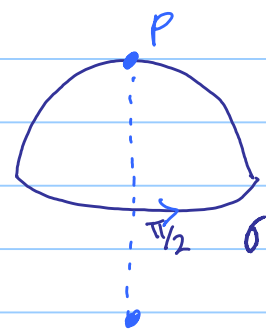
$f \geq 0$ frame function

$\operatorname{Inf}(f) = u$, $u \geq 0$

may assume $\operatorname{Inf}(f) = 0$ by subtracting u.

Let $\eta > 0$ be given.

Let $p \in S$ be such that $f(p) < \eta$, make it as the north pole.



$$\sigma = \begin{pmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} & 0 \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$x \in E$, $\langle \sigma(x) | x \rangle = 0$ (E the equator)

$\Rightarrow$ $p, x, \sigma(x)$ is a frame (orthonormal basis)

$\Rightarrow$ $f(x) + f(\sigma(x)) = W - f(p)$

Define $g(u) = f(u) + f(\sigma u)$ $u \in S$

$g|_E = W - f(p)$

g is a frame function and weight is $2W$. constant on E.

$r \in N - \{p\}$

$E_W(r) \cap E =: q$ perpendicular to r

$\Rightarrow$ $g(r) + g(q) \leq 2W$, since $q \in E$ $g(q) = W - f(p)$

So $g(r) + W - f(p) \leq 2W$

$\Rightarrow$ $g(r) \leq W + f(p) \leq W + \eta$. Since r is arbitrary in $N - \{p\}$,

$\Rightarrow$ if $x \in N - \{p\}$, then $g(x) \leq W + \eta$

Note $\quad EW(r) = S \cap \big(\overbrace{p - \langle r|p\rangle r}^{q'}\big)^{\perp}$

$r, q \in EW(r) \qquad g(r) + g(q) = 2w - g\left(\frac{q'}{\|q'\|}\right)$

let $s \in EW(r)$, $t \perp s$, $t \in EW(r)$

take $t \in N - \{p\}$

then $g(t) + g(s) = g(r) + g(q)$.

So $\quad g(r) + w - f(p) = g(s) + g(t) \leq g(s) + w + \eta$

$\Rightarrow g(r) \leq g(s) + 2\eta \qquad \forall s \in EW(r)$

Take $z \in N - \{p\}$

let U_z be the open subset of $N - \{p\}$ from the lemma

$x \in U_z$, $\exists y \in EW(x)$ s.t. $z \in EW(y)$

If $s \in EW(r)$, then $g(r) \leq g(s) + 2\eta$

Let $z \in N - \{p\}$ such that

$$g(z) < \underbrace{\operatorname{Inf}_{N-\{p\}} g}_{\beta} + \eta$$

$$\beta \leq g(x) \leq g(y) + 2\eta \leq g(z) + 4\eta \leq \beta + 5\eta$$

So $\quad \operatorname{Var}_{U_z}(g) \leq 5\eta$

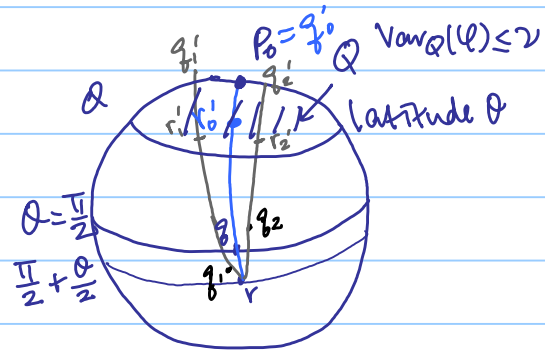
<u>Lemma</u> φ frame function.

Let $p_0 \in S$ be such that p_0 has a nbd T so that

$\operatorname{Var}_T(\varphi) = \nu$

then if $x \in$ great circle with pole p_0,

then $\exists L_x$ nbd of x such that $\operatorname{Var}_{L_x}(\varphi) \leq 2\nu$



$\langle r_0' | r \rangle = 0$

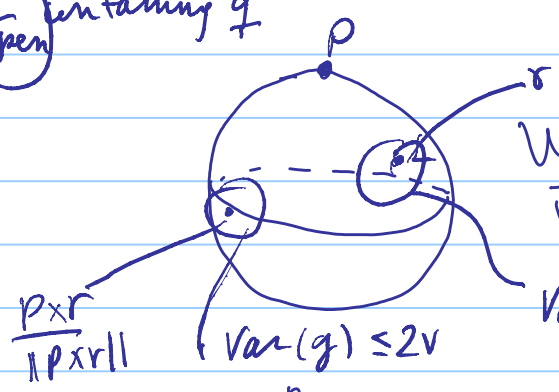
$\langle p_0 | q \rangle = 0$

$$g(r) + g(r_1') = g(q_1) + g(q_1')$$
$$g(r) + g(r_2') = g(q_2) + g(q_2')$$

$$\Rightarrow \quad g(q_1) - g(q_2) = g(r_1') - g(q_1') + g(q_2') - g(r_2')$$
$$\Rightarrow \quad |g(q_1) - g(q_2)| \le |g(r_1') - g(q_1')| + |g(q_2') - g(r_2')|$$
$$\le 2\nu$$

since all the primed elements are above latitude θ.

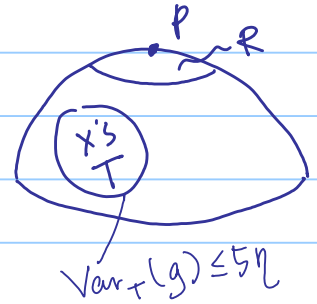
Thus there is L open containing q s.t. $\mathrm{Var}_L(g) \le 2\nu$



Under same hypothesis, if $r \in S$, $\exists U_r$ s.t. $\mathrm{Var}_{U_r}(g) \le 4\nu$

$\mathrm{Var}(g) \le 4\nu$ if $r \in N - \{p\}$

otherwise r in $N - \{p\}$.



$\exists R$ open with $p \in R$ with $\mathrm{Var}_R(g) \le 20\eta$

Know $g(p) = 2f(p)$

$\therefore \quad \mathrm{Var}_T(f) \le 22\eta$

If $r \in S$, $\exists$ nbd W_r of r such that $\mathrm{Var}_{W_r}(f) \le 88\eta$

This implies the continuity so Gleason's theorem is (finally) proved.

There was one loose end. We asserted that if $\mathcal{H}^k$ is the space of spherical harmonics of degree k and if $V \subset C(S^n)$ is a closed $O(n+1)$ invariant subspace then if $\langle V | \mathcal{H}^k|_{S^n} \rangle \neq 0$ $\mathcal{H}^k|_{S^n} \subset V$. To see this we use the fact that if $\chi_k(g) = tr(g|_{\mathcal{H}^k})$ $g \in O(n+1)$ then if $d_k = \dim \mathcal{H}^k$ then $P_k : L^2(S^n) \to \mathcal{H}^k|_{S^n}$ is given by $P_k f(x) = d_k \int_{O(n+1)} \chi_k(g) f(g^{-1}x)\, dx$.

Let $\overline{V}$ be the closure of V in $L^2(S^n)$.

Then $\langle \overline{V} | \mathcal{H}^k|_{S^n} \rangle \neq 0$ so the Riesz representation theorem implies that $\overline{V} \cap \mathcal{H}^k|_{S^n} \neq 0$. Since $O(n+1)$ acts irreducibly $\mathcal{H}^k|_{S^n} \subset \overline{V}$. Now this says that if $f \in \mathcal{H}^k|_{S^n}$ $\exists$ $f_j \in V$ such that $f_j \to f$ in $L^2(S^n)$. From the formula for P_k above we see that $P_k f_j$ is a

uniform limit of elements of V for each j. Since $P_k f_j \longrightarrow f$ in $L^2(S^n)$ we see that

$P_k f_j \in V$ and $P_k f_j \neq 0$ for j sufficiently large. Thus $V \cap \mathcal{H}^k/_{S^n} \neq \{0\}$. So irreducibility implies $\mathcal{H}^k/_{S^n} \subset V$.