Note Title 1/30/2009

$b > 1 \quad b \in \mathbb{Z}$

$$x = a_0 + \frac{a_1}{b} + \frac{a_2}{b^2} + \cdots$$

$x = a_0 . a_1 a_2 \ldots$

$0 \le a_i < b$ expansion of $x \in \mathbb{R}$ to base b.

$$x \in \mathbb{Q} \iff a_i a_{i+1} \ldots a_{r+i} . \underbrace{a_i a_{i+1} \ldots a_{r+i}} \ldots$$

$\frac{a}{b} \in \mathbb{Q}$ $\quad b > 0, \quad \gcd(a,b) = 1$

expression is unique.

$$a = a_0 \cdot b + r_0$$
$$b = a_1 \cdot r_0 + r_1$$
$$r_0 = a_2 \cdot r_1 + r_2$$
$$\vdots$$
$$r_{n-3} = a_{n-1} r_{n-2} + r_{n-1}$$
$$r_{n-2} = a_n r_{n-1} \qquad\qquad r_{n-1} = 1$$

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots \, \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}}} = \frac{a}{b}$$

$[a_0, a_1, \ldots, a_n]$

$$\frac{4537}{37}$$

$$4537 = 122 \cdot 37 + 23$$
$$37 = 1 \cdot 23 + 14$$
$$23 = 1 \cdot 14 + 9$$
$$14 = 1 \cdot 9 + 5$$
$$9 = 1 \cdot 5 + 4$$
$$5 = 1 \cdot 4 + 1$$
$$4 = 4 \cdot 1$$

$$[122, 1, 1, 1, 1, 1, 4]$$

$x \in \mathbb{R}$ x is not rational.

$$[x] = \max\{n \in \mathbb{Z} \mid n \leq x\}.$$

$$0 < x - [x] < 1 \qquad x' = \frac{1}{x-[x]}$$

$$a_0 = [x], \qquad x = a_0 + \frac{1}{x'}$$

$$1 < x', \qquad a_1 = [x'], \qquad x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{x''}}$$

$$x'' = \frac{1}{x' - a_1}$$

Yada, yada, . . .

$$[a_0, a_1, a_2, \ldots] \qquad a_0 \in \mathbb{Z}, \quad a_i > 0$$
$$i = 1, 2, \ldots$$

Partial fraction expansion of x.

$$[a_0, a_1, a_2, \ldots, a_n] = C_n .$$

Example: $\sqrt{3}$.

$$[\sqrt{3}] = 1 \qquad \frac{1}{\sqrt{3}-1} = \frac{\sqrt{3}+1}{3-1} = \frac{\sqrt{3}+1}{2} = x'$$

$$[x'] = 1 \qquad \frac{\sqrt{3}+1}{2} - 1 = \frac{\sqrt{3}-1}{2} \qquad \frac{2}{\sqrt{3}-1}$$

This yields $[1, 1, 2, \underbrace{1, 2}, \ldots]$ for $\sqrt{3}$.

We have

$a_0 = 1 \qquad a_1 = 1$

$\left\lfloor \frac{2}{\sqrt{3}-1} \right\rfloor = 2$ so $a_2 = 2$

$\frac{2}{\sqrt{3}-1} - 2 = \frac{\cancel{2}(\sqrt{3}+1)}{\cancel{2}} - 2 = \sqrt{3} - 1$

so we have $\frac{1}{\sqrt{3}-1}$ again.

Thus $a_3 = 1$, $a_4 = 2$, $a_5 = 1, \ldots$