Fix $d>0$ integer, not a square

Note Title 2/13/2009

$\exists\, \alpha > 0$ $\quad \alpha_j = \dfrac{P_j + \sqrt{d}}{Q_j}$

$\exists\, n > 0$ such

$\forall\, j > n$ then $1 \le Q_j < d$.

<u>Assert</u>: a_j, b_j so that

$\dfrac{a_j}{b_j}$ jth convergent to $\sqrt{d}$.

Looking for a convergent $\dfrac{a_j}{b_j}$ with

$a_j^2 - d b_j^2 = 1$.

First observation: If $j \geq 2$

$$a_{j-1}^2 - d\, b_{j-1}^2 = (-1)^j Q_j$$

<u>Proof:</u>

$$\sqrt{d} = \frac{a_{j-1}\alpha_j + a_{j-2}}{b_{j-1}\alpha_j + b_{j-2}} = \frac{a_{j-1}\left(\frac{P_j + \sqrt{d}}{Q_j}\right) + a_{j-2}}{b_{j-1}\left(\quad \right) + b_{j-2}}$$

Multiply top + bottom by Q_j.

$$\sqrt{d}\left(b_{j-1}(P_j + \sqrt{d}) + Q_j b_{j-2}\right) = a_{j-1}(P_j + \sqrt{d}) + Q_j a_{j-2}$$

$$\sqrt{d}\,(b_{j-1}P_j + Q_j b_{j-2}) + d\,b_{j-1} = \sqrt{d}\,a_{j-1} + a_{j-1}P_j + Q_j a_{j-2}$$

$$\text{I} \quad a_{j-1} = b_{j-1} P_j + Q_j b_{j-2}$$

$$\text{II} \quad d\, b_{j-1} = a_{j-1} P_j + Q_j a_{j-2}$$

Mult I by a_{j-1}, II by b_{j-1}

Subtract and get

$$a_{j-1}^2 - d\, b_{j-1}^2 = Q_j \left(a_{j-1} b_{j-2} - a_{j-2} b_{j-1} \right)$$

$$= (-1)^j Q_j .$$

Consider j even and $j > n$

$$a_{j-1}^2 - d\, b_{j-1}^2 = Q_j \qquad Q_j > 0 .$$

Know $1 \le Q_j < d$.

Conclusion there exists $e > 0$, $e \in \mathbb{Z}$ such that

$x^2 - dy^2 = e$ has an infinite number of solutions.

If $e = 1$ we have an infinite number of solutions. If $e > 1$.

If $x^2 - dy^2 = e$ let

$$S(x,y) = \left\{ (u,v) \;\middle|\; \begin{array}{l} u, v > 0 \text{ in } \mathbb{Z} \text{ such} \\ \text{that } u^2 - dv^2 = e \\ \text{and } u \equiv x \bmod e \\ \quad v \equiv y \bmod e \end{array} \right\}.$$

There exists (x,y) such that $S(x,y)$ is infinite.

$\exists\ (x_1,y_1) \neq (x_2,y_2) \in S(x,y)$.

write $(x_1-\sqrt{d}\,y_1)(x_2+\sqrt{d}\,y_2) = x_3+\sqrt{d}\,y_3$

$\sigma(u+\sqrt{d}\,v) = u-\sqrt{d}\,v$.

$$(x_3+\sqrt{d}\,y_3)(x_3-\sqrt{d}\,y_3) = (x_3+\sqrt{d}\,y_3)\,\sigma(x_3+\sqrt{d}\,y_3)$$

$$\overset{\shortparallel}{x_3^2-d\,y_3^2} \qquad = (x_1-\sqrt{d}\,y_1)(x_2+\sqrt{d}\,y_2)(x_1+\sqrt{d}\,y_2)(x_2-\sqrt{d}\,y_2)$$

$$= (x_1^2 - dy_1^2)(x_2^2 - dy_2^2) = e^2$$

Need is $e \mid x_3$ and $e \mid y_3$.

$$(x_1 - \sqrt{d}\, y_1)(x_2 + \sqrt{d}\, y_2) = x_3 + \sqrt{d}\, y_3$$

$$x_1 x_2 - d y_1 y_2 = x_3$$
$$x_1 y_2 - x_2 y_1 = y_3$$

consider congruence mod e

$$x_3 \equiv x^2 - dy^2 \text{ mod } e$$
$$y_3 \equiv xy - xy \text{ mod } e$$

$\Rightarrow e \mid x_3$ & $e \mid y_3$

Thus If $x = \frac{x_3}{e}$, $y = \frac{y_3}{e}$ then

$$x^2 - dy^2 = 1.$$

Replace (x, y) $(|x|, |y|)$ still a solution.

$$(x + \sqrt{d}\, y)^m = x_m + \sqrt{d}\, y_m, \quad x_m^2 - d y_m^2 = 1.$$

<u>Theorem</u>. If $x, y > 0$ give a solution of $x^2 - dy^2 = 1$ and $x + \sqrt{d}\, y$ is minimal then every solution is of this form.

Transcendental Numbers.

Louville proved existance.

A number $0 \neq \alpha \in \mathbb{R}$ is called algebraic if there exists $f(x)$ a polynomial with coefficients in $\mathbb{Q}$ such that $f(\alpha)=0$.

$\sqrt[n]{d} \longleftrightarrow x^n - d.$

$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m$

$a_i = p_i / q_i \qquad q_i > 0 \qquad q_i \in \mathbb{Z}$

Multiply by $q_0 \cdots q_m$

Then we have polynomial with coeff. in $\mathbb{Z}$.

$\alpha = r/s \quad s > 0$

$\frac{a}{b}$ rational $b > 0$

and $\alpha \neq a/b$ then $|\alpha - \frac{a}{b}| \geq \frac{1}{b}$.

Suppose α algebraic and $f(x)$ is

a polynomial of minimal degree, m, with integral coefficients such that $f(\alpha)=0$. (Note $m>1$ if α is rational).

<u>Theorem</u>. (Liouville) There exists $C>0$ If $\frac{a}{b} \in \mathbb{Q}$, $a, b \in \mathbb{Z}$, $b>0$ and α alg. and irrational with m as above (degree of α). Then $\left|\alpha - \frac{a}{b}\right| \geq \min\left(\frac{C}{b^m}, 1\right)$.

Let α be irrational of degree $m>1$. Then $\exists\, C>0$ such that if $\frac{a}{b}$ is

rational, $b>0$, $a, b \in \mathbb{Z}$ then

$$\left|\alpha - \frac{a}{b}\right| \geq \min\left(\frac{c}{b^{m_1}}, 1\right)$$