If $\alpha \in \mathbb{C}$ then we say that α is algebraic if $\exists\, f(x) = \mathbb{Q}[x]$ such that $f(x) \neq 0$ and

$$f(\alpha) = 0.$$

Examples. If $\alpha \in \mathbb{Q}$, $\exists\, f(x) = x - \alpha$.

If α is quadratic irrational then by definition $\exists\, f(x)$, $\deg f(x) = 2$ with $f(\alpha) = 0$.

$0 \neq f(x) = a_0 + a_1 x + \dots + a_r x^r \quad a_i \in \mathbb{Q}$

If $a_i = p_i/q_i$ for $a_i \neq 0$, $q_i > 0$

$\left(\prod_{a_i \neq 0} q_i\right) f(x)$ has integral coefficients.

If α algebraic and $f(x)$ is a monic polynomial (coeff of highest degree is 1) in $\mathbb{Q}[x]$ of minimal degree such that $f(\alpha) = 0$. Then $f(x)$ has following properties:

(1) It is unique. $\deg f(x) = \deg g(x)$ both monic and $f(\alpha) = g(\alpha) = 0$. Then $(f-g)(\alpha) = 0$. $\deg(f-g) < \deg f$. If $f-g$ is non-zero

then we can make it monic.

$\Rightarrow f-g=0$.

(2) If α is irrational then

$f(\beta)\neq 0$ for <u>any</u> rational number.

If β were rational root then

$f(x)=(x-\beta)\,g(x)$ with $g\in\mathbb{Q}[x]$

$0=f(\alpha)=(\alpha-\beta)\,g(\alpha) \Rightarrow g(\alpha)=0 \Rightarrow\Leftarrow$

(3) In $\mathbb{C}$, $f(x)$ has exactly

$r=\deg f(x)$ distinct roots.

<u>Proof</u>: If $f(x)$ has a double root

$r. \implies f(x) = (x-r)^2 g(x)$

$$\boxed{f'(x) = 2(x-r)g(x) + (x-r)^2 g'(x)}$$

Theorem. If α is irrational and $f(x)$, the monic polynomial of minimal degree in $\mathbb{Q}[x]$ such that $f(\alpha)=0$ has degree r. Then there exists constant $c>0$ such that if $\frac{a}{b}$ is rational with $b>0$, $a,b\in\mathbb{Z}$ then

$$\left|\alpha - \frac{a}{b}\right| \geq \min\left(\frac{c}{b^r}, 1\right).$$

Example. $\alpha = \sum_{n=1}^{\infty} \frac{1}{10^{n!}}$

$$0.11000\overset{6}{1}\underbrace{0\cdots0}\overset{24}{1}\,0\cdots\cdots0\overset{120}{1}$$

Does not satisfy Theorem for any $r > 0$ $C > 0$.

$$\sum_{n=1}^{k} \frac{1}{10^{n!}} = \frac{a_k}{\boxed{10^{k!}}} = b_k$$

$$\left| \alpha - \frac{a_k}{b_k} \right| = \sum_{n=k+1}^{\infty} \frac{1}{10^{n!}} \leq \frac{1}{10^{(k+1)!}} C$$

$$\sum_{n=k+1}^{\infty} \left(\frac{1}{10^{n!-(k+1)!}} \right) \leq C$$

$$\left|\alpha - \frac{a_k}{b_k}\right| \leq \frac{c}{b_k^{k+1}} \qquad b_k \longrightarrow \infty$$

$\Rightarrow$ α is not algebraic.

A number $\alpha \in \mathbb{C}$ that is not algebraic is called transcendental

$$a_0 + \sum_{n=1}^{\infty} \frac{a_n}{10^{n!}} \qquad a_0 \in \mathbb{Z} \text{ and } 0 \leq a_n \leq 9,\ a_n \in \mathbb{Z}$$

$\alpha \in \mathbb{R}$, $\alpha = a_0 . a_1 . . a_n . . . \longrightarrow$

irrational. Map is 1-1.

Louiville number α is a number such that $\exists\ C > 0$, $\frac{a_n}{b_n}$, $b_n \geq 0$, $b_n \to \infty$, $r_n \in \mathbb{Z}$, $r_n > 0$, $r_n \to \infty$ such that $\left|\alpha - \frac{a_n}{b_n}\right| \leq \frac{C}{b^{r_n}}$.

These are all transcendental.

Proof of the theorem. α, $f(x)$ min degree, $f(\alpha) = 0$.
multiply through by $d > 0$, $d \in \mathbb{Z}$ so that $d f(x) \in \mathbb{Z}[x]$.

Set $g(x) = d f(x)$. Observe that if $\frac{a}{b} \in \mathbb{Q}$, $b > 0$ then

$$\left|g\left(\frac{a}{b}\right)\right| \le \frac{1}{b^r} \qquad (r = \deg f(x) = \deg g(x))$$

$$g(x) = u_0 + u_1 x + \cdots + u_r x^r, \quad u_i \in \mathbb{Z}, u_r \neq 0$$

$$g\left(\frac{a}{b}\right) = \frac{1}{b^r}\Big(\underbrace{b^r u_0 + b^{r-1} u_1 a + \cdots + a^r}_{\in \mathbb{Z} - \{0\}}\Big).$$

$$\Longrightarrow \left|g\left(\frac{a}{b}\right)\right| \geq \frac{1}{b^r}.$$