Note Title 2/23/2009

If $0 \neq \alpha$ is an algebraic number and $f(x)$ is the monic polynomial in $\mathbb{Q}[x]$ of minimal degree such that $f(\alpha) = 0$ then f has $\deg f$ distinct roots in $\mathbb{C}$.

<u>Proof.</u> If f has less than $\deg f$ roots since we can factor over $\mathbb{C}$

$f(x) = \prod_{i=1}^{d} (x-\beta_i)^{m_i}$ $\beta_i \in \mathbb{C}$, $\sum_i m_i = \deg f$

$\beta_i \neq \beta_j$, $i \neq j$.. This implies that

$\exists\, \beta = \beta_i$ such that $f(x) = (x-\beta)^2 g(x)$.

$g(x) \in \mathbb{C}[x]$. $f'(x) = 2(x-\beta)g(x) + (x-\beta)^2 g'(x)$. $f'(x) \in \mathbb{Q}[x]$:

$(x^n)' = n x^{n-1}$. Hence

$\gcd(f'(x), f(x)) \neq 1$. (in $\mathbb{C}$)

But $\gcd(h_1(x), h_2(x))$ $h_i(x) \in \mathbb{Q}[x]$

is in $\mathbb{Q}[x]$. Hence $\exists\, u, v \in \mathbb{Q}[x]$

with $\deg u(x) > 0$, $\deg v(x) > 0$
and $f(x) = u(x)\, v(x)$. $\Rightarrow$
$0 = f(\alpha) = u(\alpha) v(\alpha) \Rightarrow$ say $u(\alpha) = 0$.
But $\deg u(x) < \deg f(x)$ contradicting
the <u>definition of $f(x)$</u>.

If $\alpha \in \mathbb{C}$ is an algebraic number
then $\deg \alpha$ is the degree of the monic
polynomial f_1 in $\mathbb{Q}[x]$ of minimal

degree such that $f(\alpha)=0$.

(Roots of $f(x)$ are called the conjugates of α.)

Theorem. Let α be an algebraic number of degree $m \geq 2$. Then $\exists$ $C>0$ such that if $\frac{a}{b} \in \mathbb{Q}$, $b>0$ then

$$\left|\alpha - \frac{a}{b}\right| \geq \min\left(1, \frac{C}{b^m}\right).$$

Proof. Let $f(x) \in \mathbb{Z}[x]$ of minimal degree such that $f(\alpha)=0$. $\deg f(x) = m$.

So $m \geq 2 \Rightarrow \alpha$ is irrational and this implies that $f\left(\frac{a}{b}\right) \neq 0$ for any $a/b \in \mathbb{Q}$. Let $b>0$, $a/b \in \mathbb{Q}$.

$$f(x) = a_0 + a_1 x + \cdots + a_m x^m$$

$$f\left(\frac{a}{b}\right) = a_0 + a_1 \frac{a}{b} + \cdots + a_m \left(\frac{a}{b}\right)^m$$

$$= \frac{b^m f\left(\frac{a}{b}\right)}{b^m} = \left(a_0 b^m + a_1 b^{m-1} a + \cdots + a_m a^m\right) / b^m$$

$$= \frac{u}{b^m} \qquad u \neq 0 \text{ and } u \in \mathbb{Z}.$$

$$\Rightarrow \left|f\left(\frac{a}{b}\right)\right| \geq \frac{1}{b^m}.$$

$$f(x) - f(a/b) = (x - a/b) \cdot \Big(a_1 + a_2(x + a/b) + a_3\big(x^2 + a/b\, x + \left(\tfrac{a}{b}\right)^2\big) + \cdots + a_m\big(x^{m-1} + a/b\, x^{m-2} + \cdots + (a/b)^{m-1}\big)\Big)$$

$$(x-y)(x^k + x^{k-1}y + \cdots + y^k)$$

$$= x^k - y^k.$$

$$= \left(x - \frac{a}{b}\right) g(x) \quad , \; g(x) \in \mathbb{Q}[x].$$

Suppose $|\alpha - a/b| \geq 1$ inequality is obvious. May assume that $|\alpha - a/b| \leq 1$

$\Rightarrow \quad |a/b| \leq 1 + |\alpha|$. Consider

$$g(\alpha) = a_1 + a_2(\alpha + a/b) + a_3(\alpha^2 + a/b\,\alpha + (a/b)^2)$$

$$+ \cdots + a_m\left(\alpha^{m-1} + a/b\,\alpha^{m-2} + \cdots + \left(\frac{a}{b}\right)^{m-1}\right)$$

$$|g(\alpha)| \leq |a_1| + |a_2|\big(|\alpha| + (1+|\alpha|)\big) +$$

$$|a_3|\left(|\alpha|^2+(1+|\alpha|)|\alpha|+(1+|\alpha|)^2\right)$$

$$+\ \cdots$$

$$\le D \qquad , \quad D>0.$$

$$\underbrace{\underbrace{|f(\alpha)}_{\substack{\shortparallel \\ 0}}-f(\tfrac{a}{b})|}_{\shortparallel} = |\alpha-\tfrac{a}{b}|\cdot|g(\alpha)| \le |\alpha-\tfrac{a}{b}|D$$

$$|f(\tfrac{a}{b})|$$

$$\geqslant \frac{1}{b^m} \qquad \Rightarrow \qquad |\alpha-a/b| \geqslant \left(\frac{1}{D}\right)\frac{1}{b^m}$$

$$C=\frac{1}{D}.$$

QED.

<u>Ex:</u> p. 161 8.

1873 Hermite proved that
e is transcendental.
Lindemann proved π is also.

$e^e \quad e^\pi \quad \pi^e$

$e^\alpha \quad \alpha$ algebraic $\alpha \neq 0$
then e^α is transcendental.
Note $e^{i\pi} = -1 \Rightarrow i\pi$ is transcendental

$\Rightarrow \pi$ is also.