Note Title 2/25/2009

Ex. (1) If $\gamma \in \mathbb{C}$ is transcendental then γ^α is transcendental if $\alpha \in \mathbb{Q}$ and $\alpha \neq 0$.

(2) If $\gamma \in \mathbb{C}$ is transcendental and α, β are rational $\alpha \neq 0$ then $\gamma^\alpha + \beta\gamma^{-\alpha}$ is transcendental.

$\Rightarrow$ if $\alpha \in \mathbb{Q}$, $\alpha \neq 0$ then $e^\alpha + \beta e^{-\alpha}$ is transcendental since we prove that e is $\mathcal{T}$.

(Hermite, 1873)

Theorem. e is transcendental.

Proof. By contradiction. Assume there exist $a_0, a_1, \dots, a_n \in \mathbb{Z}$ with $a_0 + a_1 e + \dots + a_n e^n = 0$ and $a_0, a_n \neq 0$. Find a contradiction.

$$(*) \quad f = f_{p,n}(x) = x^{p-1} \prod_{j=1}^{n} (x-j)^p.$$

p a prime and $p > a_0$ and n.

$$(**) \quad H(x) = e^x \sum_{j=0}^{n} f^{(j)}(0) - \sum_{j=0}^{n} f^{(j)}(x)$$

$$(***) \quad L = \sum_{j=0}^{n} a_j H(j).$$

We show there exist $C>0$, $b>0$ independent of p so that

$$(p-1)! \leq |L| \leq C b^p$$

Observe that $\lim_{p \to \infty} \frac{b^p}{(p-1)!} = 0.$

Contradiction.

$\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges for all $x \in \mathbb{R}$

$$L = \sum a_j H(j) = \underbrace{\left(\sum a_j e^j\right)}_{\overset{\parallel}{0}} \left(\sum_{r=0}^{n} f^{(r)}(0)\right)$$

$m = \deg f = (n+1)p - 1$

$$- \sum a_j \sum_{i=1}^{m} f^{(i)}(j)$$

$$L = - \sum_{j,i} a_j f^{(i)}(j) \in \mathbb{Z}.$$

We ~~now~~ prove

$L \equiv 0 \mod (p-1)!$

$L \not\equiv 0 \mod p.$

p is prime.

$p > a_0, p > n.$

$\Rightarrow |L| \geq (p-1)!$ (Come back to this)

Consider $f_{p,n}(x) = x^{p-1}(x-1)^p(x-2)^p \cdots (x-n)^p$

assume $f(x)$ arbitrary polynomial of deg m:

$$H_f(x) = e^x \sum_{j=0}^{m} f^{(j)}(0) - \sum_{j=0}^{m} f^{(j)}(x)$$

We prove

Lemma. $|H_f(x)| \leq e^{|x|} \sum |c_j||x|^j$

where $f(x) = \sum_{j=0}^{m} c_j x^j$.

Proof. Manipulation. (Put off)

$$f_{p,n}(x) = x^{p-1}(x-1)^p \cdots (x-n)^p$$

$$f_{p,n}(-x) = (-1)^{p-1} \cdot (-1)^{np} \Big(\underbrace{x^{p-1}(x+1)^p \cdots (x+n)^p}_{u(x)} \Big)$$

If $f_{p,n}(x) = \sum c_{p,n,j} x^j$

and $f_{p,n}(-x) = (-1)^{p-1}(-1)^{np} \sum d_{p,n,j} x^j$

$$|c_{p,n,j}| \leq d_{p,n,j}.$$

$$f_{p,n}(x) = x^{p-1} \sum{}' \binom{p}{l_1}(-1)^{p-l_1} x^{l_1} \binom{p}{l_2}(-1)^{p-l_2} 2^{p-l_2} x^{l_2}$$

$$\cdots \binom{p}{l_n}(-1)^{p-l_n} x^{l_n} n^{p-l_n}$$

$$x^{p-1} \sum{}' \binom{p}{l_1}\binom{p}{l_2}\cdots\binom{p}{l_n}(-1)^{np-l_1-\cdots-l_n} 1^{p-l_1} 2^{p-l_2} \cdots n^{p-l_n} x^{l_1+\cdots+l_n}$$

$$C_{p,n,j+p-1} = \sum_{l_1+\cdots+l_n=j}$$

$$|C_{p,n,j+p-1}| \leq$$

without signs

$0 \le j \le n$ that is $d_{p,n,j+p-1}$.

$$|H(j)| \le \underbrace{e^{j} \sum_{i=1}^{j:m} |c_{p,n,j}| \, j^{i}}_{u(j)} \le e^{j} j^{p-1} \prod_{i=1}^{n} (j+i)^{p} \quad j+i \le 2n$$

$$\le e^{j}$$

$$\le e^{j} (2n)^{(n+1)p} \quad \text{set } b = (2n)^{n+1}$$

$$|H(j)| \le e^{j} b_{n}^{p} \quad \text{So}$$

$$L_i = \sum_{j=0}^{n} a_j H(j) \Rightarrow$$

$$|L| \leq \sum_{j=0}^{n} |a_j| \, |H(j)| \leq \Big(\overbrace{\sum_{j=0}^{n} |a_j| e^{j}}^{C}\Big) b^p$$

$$C = \sum_{j=0}^{n} |a_j| e^{j}.$$ Here

$$(p-1)! \leq |L| < C b^p$$

$$\sum_{j=0}^{n} a_j \sum_{i=0}^{n} f^{(i)}(j) \; -$$

$1 \le j \le n$ $\quad (x-j)^p$ a factor

$f^{(i)}(j) \neq 0 \Rightarrow i \geq p$ so this is divisible by $p!$

Left with $a_0 \sum_{i=0}^{m} f^{(i)}(0)$

i.e. $\sum_{j=0}^{n} a_j f^{(i)}(j) \equiv$ $\uparrow$ mod $p!$

$0 \le i \le m$

$f^{(i)}(0) \neq 0 \Rightarrow i \geq p-1$

if $i = p-1$, $(p-1)! \prod_{j=1}^{n} (-j)^p = f^{(p-1)}$

if $i > p$ wind up with extra p.

$$\Rightarrow \sum_{j=0}^{n} a_j f^{(j)}(j) \equiv a_0 (p-1)! \prod_{i=1}^{n} (-i)^p \mod \boxed{p} \quad \text{n<p} \ \text{~~small~~}$$

$$\wedge \quad \text{p}$$

$\Rightarrow$ assertion of divisibility

by $(p-1)!$ and that

the expression is not $= 0$.