Note Title 2/9/2009

Suppose $y < 0$

$b_k x + b_{k+1} y = S$

$\Rightarrow b_k x = S - b_{k+1} y > 0$

$\Rightarrow x > 0$

If $y > 0$

$b_{k+1} y \geq b_{k+1} > S \Rightarrow b_k x = S - b_{k+1} y$
$< 0.$

$$\frac{a_{k+1}}{b_{k+1}} < \alpha < \frac{a_k}{b_k} \quad \text{or} \quad \frac{a_k}{b_k} < \alpha < \frac{a_{k+1}}{b_{k+1}}$$

$$\Rightarrow (b_k \alpha - a_k)(b_{k+1}\alpha - a_{k+1}) < 0$$

$b_k > 0$ all k.

Theorem. If $r, s \in \mathbb{Z}$, $s > 0$, and α irrat.

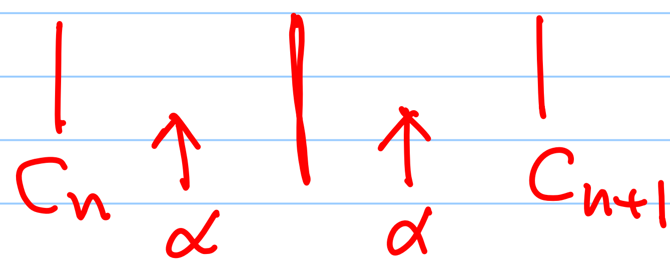
$$\left| \frac{r}{s} - \alpha \right| < \frac{1}{2s^2} \Rightarrow \frac{r}{s} \text{ is}$$

is a convergent of α.

$$C_n < \alpha < C_{n+1}$$

$$|C_n - \alpha| < |C_n - C_{n+1}| = \frac{1}{b_n b_{n+1}} \le \frac{1}{b_n^2}$$

$$b_{n+1} = b_n q_{n+1} + b_{n-1}.$$



Proof: assume $\frac{r}{s} \neq C_k$

for any $k = 1, 2, \ldots$

$\therefore$ $\exists\, k$ such that

$$b_k \leq s < b_{k+1}$$

Have $|s\alpha - r| \geq |b_k \alpha - a_k|$

$$\Rightarrow b_k \left|\alpha - \frac{a_k}{b_k}\right| \leq s \left|\alpha - \frac{r}{s}\right|$$

$$\Rightarrow b_k \left|\alpha - \frac{\alpha_k}{b_k}\right| < \frac{1}{2s}$$

$$\Longrightarrow \quad \left|\alpha - \frac{a_k}{b_k}\right| < \frac{1}{2b_k s}.$$

$$\frac{r}{s} \neq \frac{a_k}{b_k} \Longrightarrow \left|\frac{r}{s} - \frac{a_k}{b_k}\right| = \frac{|b_k r - a_k s|}{s b_k}$$

$$\geqslant \frac{1}{s b_k} \qquad \frac{1}{s b_k} \leqslant \left|\frac{r}{s} - \frac{a_k}{b_k}\right|$$

$$= \left|\frac{r}{s} - \alpha + \alpha - \frac{a_k}{b_k}\right| \leq$$

$$\left|\frac{r}{s} - \alpha\right| + \left|\alpha - \frac{a_k}{b_k}\right|$$

$$\frac{1}{s b_k} < \frac{1}{2s^2} + \frac{1}{2b_k s} \Rightarrow$$

$$\frac{1}{2b_k s} < \frac{1}{2s^2} \Rightarrow s < b_k \Rightarrow\Leftarrow$$

since choose k so that $b_k \leq s < b_{k+1}$

Theorem. α is a quadratic irrational $\Leftrightarrow$ its simple continued fraction decomposition is periodic.

$\in \mathbb{Z}, d > 0$

If d is not square then there are an infinite number of integral solutions to

$$x^2 - dy^2 = 1 \quad *$$

Pell's Equation.

$$x^2 - 12y^2 = m.$$

If (x, y) $x, y > 0$ is a non-trivial solution to $(*)$ then $\frac{x}{y}$ is a convergent

$x = a_n, \; y = b_n$

$$1 = x^2 - dy^2 = (x-\sqrt{d}\,y)(x+\sqrt{d}\,y)$$

$$(x-\sqrt{d}\,y) = \frac{1}{x+\sqrt{d}\,y}$$

$$\Rightarrow \left|\frac{x}{y} - \sqrt{d}\right| = \frac{1}{y(x+\sqrt{d}\,y)} < \frac{1}{2y^2}$$

$$x^2 > dy^2 \Rightarrow x+\sqrt{d}\,y > 2y \Rightarrow$$

$d=2.$ $\quad \sqrt{2} = (1,2,2,2,\dots)$

$\frac{1}{1}$ $\quad 1+\frac{1}{2} = \frac{3}{2}$ $\quad 3^2 - 2\cdot 2^2 = 1$

$$7^2 - 2\cdot 5^2 = -1.$$

Lemma. If $x_1^2 - d y_1^2 = 1$

$x_2^2 - d y_2^2 = 1$

Then if $x_3 + \sqrt{d}\, y_3 = (x_1 + \sqrt{d}\, y_1)(x_2 + \sqrt{d}\, y_2)$

Then

$x_3^2 - d y_3^2 = 1.$

$x_k - \sqrt{2}\, y_k$

$d = \sqrt{2}$ $\quad (3 + \sqrt{2}\, 2)^k = x_k + \sqrt{2}\, y_k$

$x_k^2 - 2 y_k^2 = 1.$ $\quad ((3 + \sqrt{2}\, 2)(3 - \sqrt{2}\, 2))^k = 1$