Note Title 3/2/2009

Assignment for today in Web.

Chebyshev: If $\pi(x)$ is the number of primes p such that $1 \le p \le x$ then

$$\frac{1}{10}\frac{x}{\log x} < \pi(x) < 10\frac{x}{\log x}$$

1850

Given $\varepsilon > 0, \delta > 0$ $\exists$ N such that if $x \geq N$

$$(1-\delta)\frac{x}{\log x} < \pi(x) < (1+\varepsilon)\frac{x}{\log x}$$

$$\lim_{x \to \infty} \frac{\pi(x)}{\frac{x}{\log x}} = 1. \qquad \underline{\underline{\text{PNT}}} \sim 1800 \text{ Conj.}$$

$$* \qquad n^{\pi(2n)-\pi(n)} \leq \binom{2n}{n} \leq (2n)^{\pi(2n)}$$

Use in form:

$$n^{\pi(2n)-\pi(n)} \leq 2^{2n}, \qquad 2^{n} \leq (2n)^{\pi(2n)}$$

Must show $\binom{2n}{n} \leq 2^{2n}$

$\binom{2n}{n} \geq 2^{n}$

$$2^{2n} = (1+1)^{2n} = \sum_{j=0}^{2n} \binom{2n}{j} > \binom{2n}{n}$$

$$\frac{2n}{n} \cdot \frac{2n-1}{n-1} \cdots \cancel{3}\binom{n}{1} \geq 2^{n} \qquad n \geq 2$$

$$\begin{array}{cc} \frac{2n}{n} & \frac{2n-1}{n-1} \\ \wedge\!| & \wedge\!| \\ 2 & 2 \end{array}$$

$$2^{n} \leq \cancel{\#}(2n)^{\pi(2n)}$$

$$(\log 2)\, n \leq \log(2n)\, \pi(2n)$$

$$\pi(2n) \geq \left(\frac{\log 2}{2}\right) \frac{2n}{\log(2n)}$$

We show that

$$\pi(x) \geq \frac{\log 2}{8} \; \frac{x}{\log x}$$

if x even integer

$$\pi(x) \geq \left(\frac{\log 2}{2}\right) \frac{x}{\log x}$$ if x is

an odd integer then

If x isn't a prime then $\pi(x) = \pi(x-1)$

If x is a prime then $\pi(x) = \pi(x-1)+1$

$$\pi(x) \geq \left(\frac{\log 2}{2}\right)\frac{x-1}{\log(x-1)} + 1 \geq \left(\frac{\log 2}{2}\right)\frac{x-1}{\log(x-1)}$$

$x-1 \geq x/2$ if $x \geq 2$

$$\Rightarrow \pi(x) \geq \frac{\log 2}{4}\frac{x}{\log x} \qquad x \in \mathbb{Z},\ x \geq 2$$

Must do same for $x \in \mathbb{R},\ x \geq 2$

$$\pi(x) = \pi(\lceil x\rceil) \geq \frac{\log 2}{4} \cdot \frac{\lceil x\rceil}{\log\lceil x\rceil} \geq \frac{\log 2}{4}\frac{\lceil x\rceil}{\log x} \geq \frac{\log 2}{8}\left(\frac{x}{\log x}\right).$$

$$\overline{\pi(x) \geq \frac{\log 2}{8}\left(\frac{x}{\log x}\right).}$$

$$n^{\pi(2n)-\pi(n)} \le 2^{2n}$$

set $n = 2^{k-1}$

$$2^{(k-1)(\pi(2^k)-\pi(2^{k-1}))} \le 2^{2^k}$$

$$\Rightarrow \quad (k-1)\left(\pi(2^k)-\pi(2^{k-1})\right) \le 2^k.$$

$$k\pi(2^k) \le \pi(2^k) + (k-1)\pi(2^{k-1}) + 2^k$$

$$\pi(2^k) \le \cancel{\pi(2)}\ 2^{k-1}. \qquad k \ge 1.$$

$\pi(2^{1}) = 2 = 2^1.$ $\qquad k=3 \quad \pi(2^3) = \pi(8) = 4$

$1 \quad 2 \quad \cdot \quad \cdot \quad 2^k$ $\qquad$ half are even. $\quad k \geq 4$

$2^{k-1} - 1$ $\qquad 2^{4} = 16 \quad 9$

$$\pi(2^k) \geq 2^{k-1} \text{ all } k \geq 1.$$

$$k\pi(2^k) \leq (k-1)\pi(2^{k-1}) + 2^{k-1} + 2^{k-1}\cdot 2.$$

$$\leq (k-1)\pi(2^{k-1}) + 3 \cdot 2^{k-1}$$

Want $\quad k\pi(2^k) \leq \lambda 2^k.$

$$(k-1)\pi(2^{k-1}) \leq \lambda 2^{k-1}$$

$$k\pi(2^k) \le (\lambda+3)2^{k-1} = \frac{(\lambda+3)}{2}\cdot 2^k$$

$$\frac{\lambda+3}{2} \le \lambda \quad \text{or} \quad \lambda \ge 3.$$

$$\Rightarrow \quad k\pi(2^k) \le 3\cdot 2^k. \quad \Rightarrow \quad \pi(2^k) \le \frac{3\cdot 2^k}{k}$$

Consider $2^k \le x < 2^{k+1}$

$$\log x \;\; \pi(x) \le (\log 2)(k+1)\,\pi(2^{k+1}) \le 3\cdot 2^{k+1}(\log 2) \le 6\cdot 2^k(\log 2) \le 6x\log 2.$$

Modulo $(*)$ have shown

$$\frac{\log 2}{8}\frac{x}{\log x} \le \pi(x) \le 6(\log 2)\frac{\log x}{x} \quad \text{for } x \ge 2.$$

$(*)$ $$\binom{2n}{n} \geq n^{\pi(2n)-\pi(n)}$$

$$\binom{2n}{n} \leq (2n)^{\pi(2n)}$$

$$\binom{2n}{n} = \frac{(2n)!}{(n!)^2} \qquad \text{if } n+1 \leq p \leq 2n$$

p divides numerator but not the denominator.

$n < p \leq 2n$, $\quad p^k \mid (2n)!$ then $p^k \mid \binom{2n}{n}$.

$$\prod_{n<p\leq 2n} p \perp \binom{2n}{n} \Longrightarrow$$

$$\binom{2n}{n} \geq \prod_{n<p\leq 2n} p \geq n^{\pi(2n)-\pi(n)}$$

$$\binom{2n}{n} \leq (2n)^{\pi(2n)}.$$

Need more information on primes.

Lemma. If $n! = \prod_{2 \le p \le n} p^{e_p}$ product of distinct primes. Then

$$e_p = \sum_{j \geq 1} \left[\frac{n}{p^j} \right].$$

Suppose we consider all multiples of $p^j \le n$, $j > 0$, p prime

$p^j, 2 \cdot p^j, \ldots, r_j p^j$ if $p^j \le n$

But $(r_j + 1) p^j > n$.