Note Title 3/4/2009

$$e_p = \sum_{j=1}^{\lfloor \log_p(n) \rfloor} \left\lfloor \frac{n}{p^j} \right\rfloor .$$

<u>Need</u> $\binom{2n}{n} \leq (2n)^{\pi(2n)}$.

$\binom{2n}{n} = \frac{(2n)!}{n!\,n!}$ if r_0 is the highest power of p a prime dividing

$\binom{2n}{n}$ then $r_0 = \sum_{j=1}^{\lceil \log_p(2n) \rceil} \left(\left[\frac{2n}{p^j}\right] - 2\left[\frac{n}{p^j}\right] \right)$

Note: If $a > 0$ then $[2a] - 2[a] \in \{0,1\}$.

$a = [a] + b$ with $0 \le b < 1$

$2a = 2[a] + 2b \Rightarrow [2a] = \begin{cases} 2[a] & \text{if } 0 \le b < \frac{1}{2} \\ 2[a]+1 & \text{if } \frac{1}{2} \le b < 1 \end{cases}$

$\Rightarrow r_0 \le [\log_p(2n)] \Rightarrow p^{r_0} \le p^{\log_p(2n)}$

$\Rightarrow p \le 2n.$

$$\binom{2n}{n} = \prod_{1 \le p \le 2n} p^{r_0(p)} \le (2n)^{\pi(2n)}.$$

Suppose $p^j \le n$

$p^j, 2\cdot p^j, \ldots, r_j \cdot p^j$ (where $r_j p^j \le n$ and $(r_j+1)p^j > n$) are the numbers between 1 and n that are divisible by p^j. Hence number of $1 \le m \le n$ such that $p^j \mid m$ but $p^{j+1} \nmid m$

is $\quad r_j - r_{j+1}$.

$$\Rightarrow \quad e_p = \sum_{j=1}^{\log_p(n)} j(r_j - r_{j+1}) \quad \text{but}$$

$$r_j = \left\lceil \frac{n}{p^j} \right\rceil \longrightarrow e_p = \sum_{j=1}^{\log_p(n)} j\left(\left\lceil \frac{n}{p^j} \right\rceil - \left\lceil \frac{n}{p^{j+1}} \right\rceil\right)$$

$$\left\lceil \frac{n}{p} \right\rceil - \left\lceil \frac{n}{p^2} \right\rceil + 2\left(\left\lceil \frac{n}{p^2} \right\rceil - \left\lceil \frac{n}{p^3} \right\rceil\right) + 3\left(\left\lceil \frac{n}{p^3} \right\rceil - \left\lceil \frac{n}{p^4} \right\rceil\right)$$

$$+ \ldots = \sum_{j \geq 1} \left\lceil \frac{n}{p^j} \right\rceil.$$

$$\log n! = \sum_{j,p} (\log p) \left[\frac{n}{p^j}\right].$$

$$\log n! = \sum_{r=1}^{n} \log r$$

$$(.9)\frac{\log x}{x} < \pi(x) < 1.2\frac{\log x}{x}$$

if x sufficiently large.

PNT: $$\lim_{x\to\infty} \frac{\pi(x)\log x}{x} = 1$$

$$\mathrm{Li}(x) = \int_2^x \frac{dt}{\log t}$$

$$\frac{d}{dx} \frac{x}{\log x} = \frac{\log x - 1}{(\log x)^2} = \frac{1}{\log x} - \frac{1}{(\log x)^2}$$

$$\frac{d}{dx} \mathrm{Li}(x) = \frac{1}{\log x}$$

$\pi(x) \sim \mathrm{Li}(x)$ Conjecture of Gauss

$\mathrm{Li}(x) > \pi(x)$ for $x > 3$.

$\sim 10^{316}$. $\underline{(2000)}$ $10^{10^{100}}$

Riemann Zeta Function.

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}. \quad \text{If } s = \sigma + i\tau$$

$\sigma, \tau \in \mathbb{R}$

$|n^s| = n^\sigma$. Integral test $\Rightarrow$

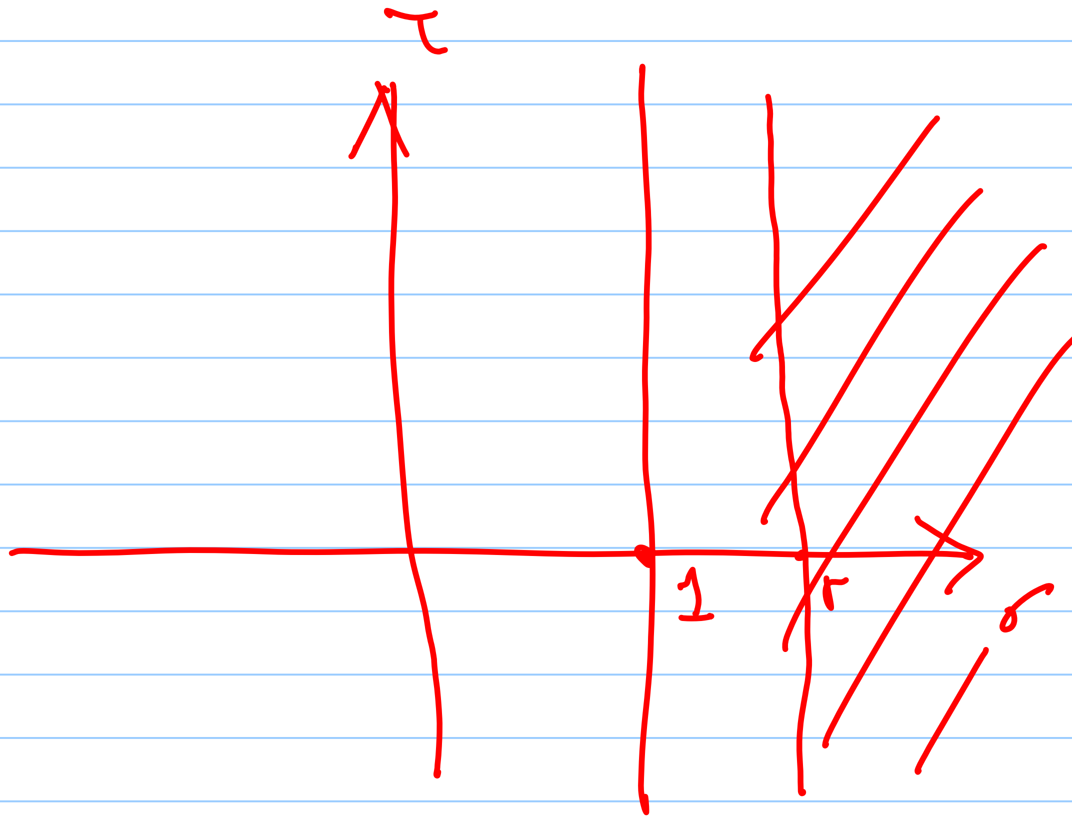
If $\sigma > 1$ then $\sum_{n=1}^{\infty} \frac{1}{n^\sigma}$ converges

$$\left|\zeta(s) - \sum_{n=1}^{N}{}' \frac{1}{n^s}\right| \leq \sum_{n=N+1}^{\infty}{}' \frac{1}{n^r} \quad \text{if}$$

$\operatorname{Re} s \geq r > 1.$

$\downarrow$

0 as $N \to \infty$



Math 120

$\Downarrow$

$\zeta(s)$ is complex

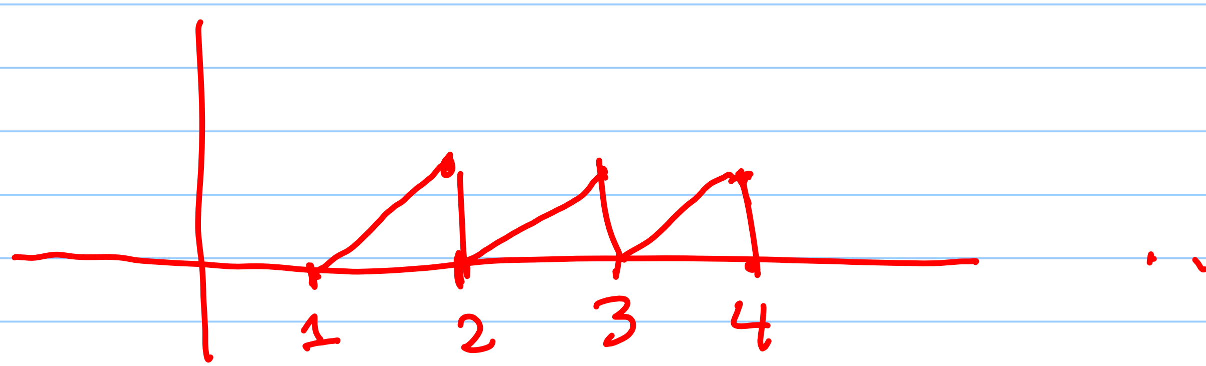
analytic for $\operatorname{Re} s > 1$.

Riemann observed:

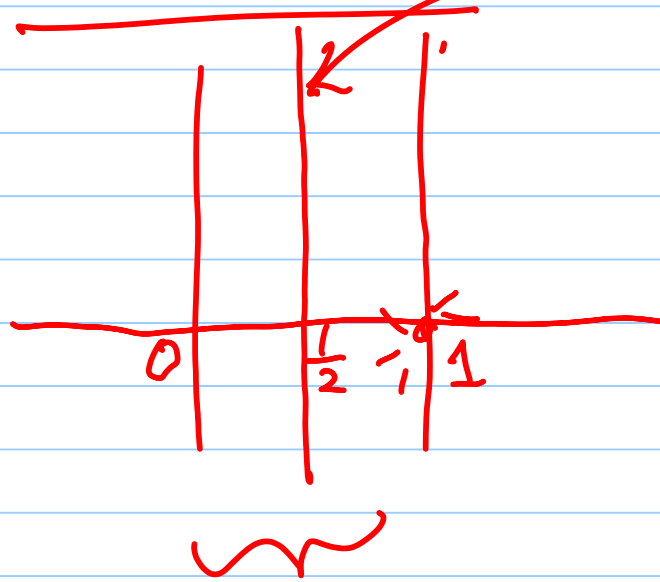
$$\zeta(s) = \frac{1}{s-1} + 1 - s\int_1^{\infty} \frac{x - [x]}{x^{s+1}}\, dx$$

$\operatorname{Re} s > 0$

Graph $x - [x]$ $\quad x \geq 1$



Integral: $\displaystyle\sum_{n=1}^{\infty} \int_0^1 \frac{t}{(n+t)^{s+1}}\,dt$



Riemann hypothesis says the only zeros of $\zeta(s)$ for $0 < \operatorname{Re} s < 1$ are for $\operatorname{Re} s = \frac{1}{2}$.

PNT is the assertion that there are no zeros on $\mathrm{Re}\, s = 1$.

Can prove $C' > 0$, $C > 0$

$$C'(x^{1/2}\log x) \leq \pi(x) - \mathrm{Li}(x) \leq C(x^{1/2} \cdot \log x)$$

Then Riemann hyp. follows.

$$\zeta(2) = \frac{\pi^2}{6} \qquad \text{(Euler)}$$

$\zeta(2n)$ $\quad n = 1, 2, \ldots$

$$\zeta_N(s) = \prod_{\substack{2 \le p \le N \\ p \text{ prime}}} \frac{1}{1-p^{-s}} \qquad \mathrm{Re}\, s > 1$$

$$0 < |p^{-s}| < 1 . \qquad |z| < 1$$

$$\frac{1}{1-z} = \sum_{n=1}^{\infty} z^n$$

$$\frac{1}{1-p^{-s}} = \sum_{n=0}^{\infty} p^{-ns}$$

Let $p_1, \dots, p_m$ be the primes $2 \le p_i \le N$

$m = \pi(N)$.

$$\prod_{i=1}^{m} \frac{1}{1-p_i^{-s}} = \sum_{e_1 \geq 0, e_2 \geq 0, \ldots, e_m \geq 0} \left(p_1^{e_1} \cdot p_2^{e_2} \cdots p_m^{e_m}\right)^{-s}$$

$$= \sum_{n=1}^{\infty} a_n \cdot n^{-s} \quad \text{with}$$

$$a_n = \begin{cases} 1 & \text{if } n = p_1^{e_1} \cdots p_m^{e_m}, e_i \geq 0 \\ 0 & \text{if } n \text{ is not of this form.} \end{cases}$$

$$= \sum_{1 \le n \le N} n^{-s} + \sum_{N < m < \infty} a_m n^{-s}$$

$\operatorname{Re} s > 1$

$$\left| \prod_{i=1}^{m} \frac{1}{1 - p_i^{-s}} - \zeta(s) \right| \le \sum_{n > N} n^{-\operatorname{Re} s} \le \sum_{n > N} n^{-\sigma}$$

if $\operatorname{Re} s > \sigma > 1$.

$$\lim_{N \to \infty} \frac{1}{\prod_{2 \le p \le N} (1 - p^{-s})} = \zeta(s) \text{ if } \operatorname{Re} s > 1.$$

$$\zeta(s) = \frac{1}{\prod_p (1-p^{-s})}$$

$$\Gamma(s) = \int_0^\infty e^{-x} x^{s-1} dx =$$

$\mathrm{Re}\, s > 0$. x^{σ}, $\sigma > -1$ is integrable on $[0,1]$.

$$\int_0^1 x^{\sigma} dx = \frac{1}{\sigma+1} x^{\sigma+1} \Big|_0^1$$

$$\Gamma(s) = \lim_{N\to\infty} \int_0^N e^{-x} x^{s-1}\,dx$$

$$x^{s-1} = \frac{1}{s}\frac{d}{dx}x^s$$

$$\int_0^N e^{-x} x^{s-1}\,dx = \frac{1}{s}\int_0^N e^{-x}\frac{dx^s}{dx}\,dx$$

$$\Gamma(s) = \lim_{N\to\infty} \underbrace{\left(e^{-x}x^s\right)\Big|_0^N}_{= 0} + \frac{1}{s}\int_0^\infty e^{-x}x^s\,dx$$

$\operatorname{Re} s > 0$

$\Rightarrow$

$$s\Gamma(s) = \Gamma(s+1).$$

$$\Gamma(2) = 1\cdot\Gamma(1) = \int_0^\infty e^{-x}\,dx = 1.$$

$$\Gamma(3) = 2\cdot\Gamma(1) = 2$$

$$\Gamma(4) = 3\cdot\Gamma(3) = 3\cdot 2$$

Sterling's formula.

$$\Gamma(n+1) = n!$$

$$n! \sim n^n\cdot e^{-n}\cdot(\pi n)^{1/2}$$

$$\log n! = \sum_{i>0,\ p \text{ prime}} (\log p)\left[\frac{n}{p^i}\right]$$

$$\tilde{\zeta}(s) = \pi^{-s/2}\,\Gamma(s/2)\,\zeta(s)$$

$\mathbb{Q}_p$, $|x+y|_p \leq |x|_p + |y|_p$, $|xy|_p = |x|_p|y|_p$

$\mathbb{R}$, $|x|_\infty = |x|$

$2 \leq p < \infty$

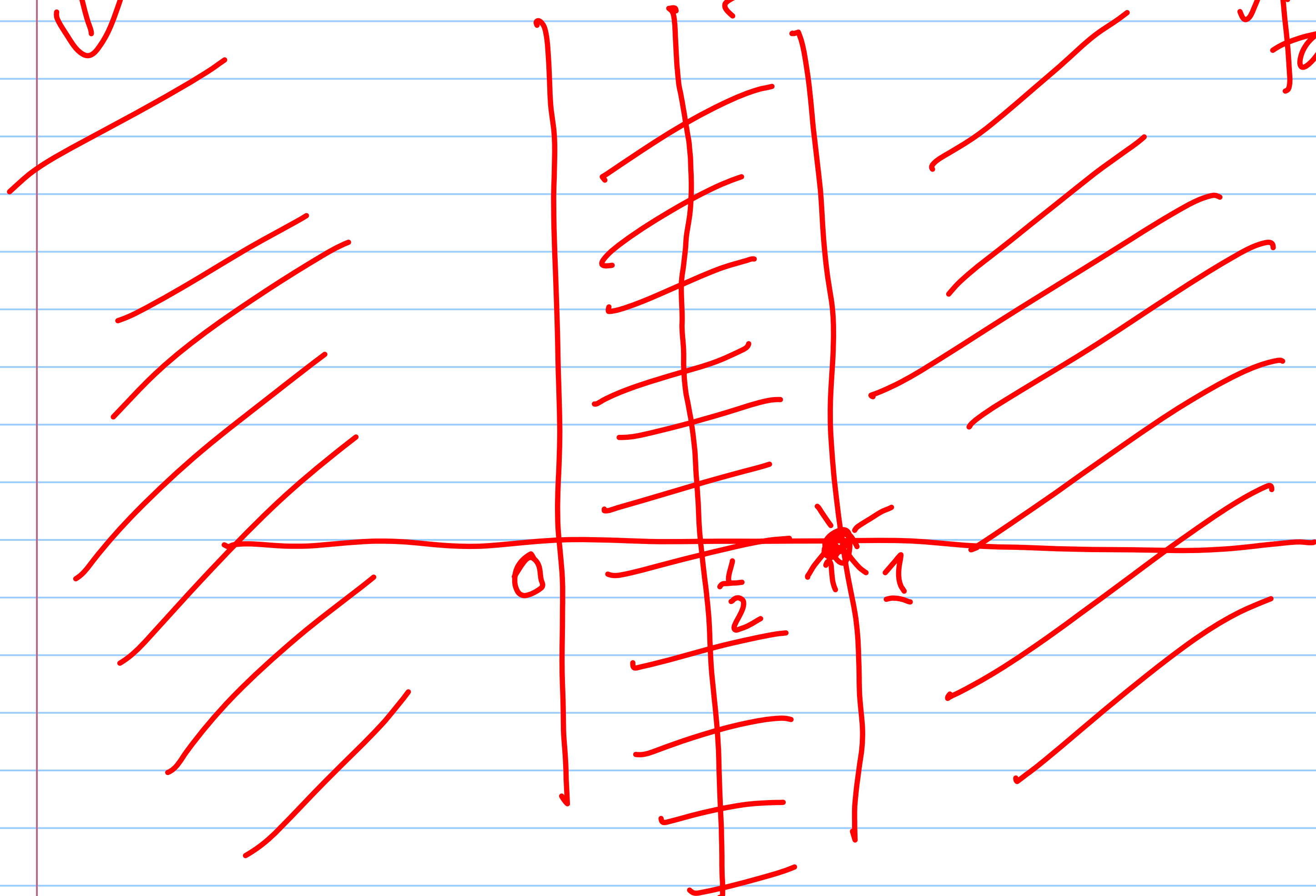
$p \longrightarrow \dfrac{1}{1-p^{-s}}$ $\quad \infty \longrightarrow \pi^{-s/2}\,\Gamma(s/2)$

Functional equation

defines here

$\tilde{\zeta}(1-s) = \tilde{\zeta}(s)$

using alternate formula



convergence of $\sum_{n \geq 1} n^{-s}$

1. What is the value of $\zeta(6)$?

(Hint: zeta(s) is a function in Pari).

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$ Euler

$$\zeta(2n) = \pi^{2n} \cdot c_n$$ Euler

$c_n \in \mathbb{Q}$. c_n?

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

$$\frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)}$$

$$\left(1-\frac{1}{2}\right) + \left(\frac{1}{2}-\frac{1}{3}\right) + \left(\frac{1}{3}-\frac{1}{4}\right) + \cdots = 1$$

2. Use leibniz method to sum

$$\sum_{n=1}^{\infty} \frac{1}{\binom{n+k-1}{k}}$$

(Hint: Prove

$$\frac{1}{k\binom{n+k-1}{k}} - \frac{1}{k\binom{n+k}{k}} = \frac{1}{(k+1)\binom{n+k}{k+1}}$$

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