Note Title 3/9/2009

Final Friday Here

Euler $\zeta(2n)$ $n \in \mathbb{Z}$, $n \geq 1$.

$\zeta(0) = ?$ $\zeta(2) = \frac{\pi^2}{6}$.

Suppose $f(t) \in \mathbb{C}[t]$. $\deg f(t) = n$

We assume $f(0) \neq 0$.

$$f(t) = \prod_{i=1}^{n} (t - r_i) \qquad r_1, \dots, r_n$$

roots counted with multiplicity.

$$f(t) = (-1)^n \left(\prod r_i\right) \cdot \prod_{i=1}^{n} \left(1 - \frac{t}{r_i}\right)$$

$f(0) = (-1)^n \left(\prod r_i\right)$. Divide by this.

Get $g(t) = \prod_{i=1}^{n} \left(1 - \frac{t}{r_i}\right)$, $g(0) = 1$.

$$g(t) = 1 - \left(\frac{1}{r_1} + \cdots + \frac{1}{r_n}\right)t + \left(\sum_{i<j} \frac{1}{r_i r_j}\right)t^2$$

$$- \cdots$$

$$g(t) = 1 + a_1 t + a_2 t^2 + \cdots + a_n t^n$$

$$a_1 = -\sum_{i=1}^{n} \frac{1}{r_i}.$$

$$\sin x = \sum (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\frac{\sin x}{x} = \sum_1 (-1)^n \frac{x^{2n}}{(2n+1)!}$$

$$\frac{\sin(\pi n)}{\pi n} = 0 \qquad n = \cancel{0}, 1, \ldots$$

Replace x^2 by t.

$$g(t) = \sum_1 (-1)^n \frac{t^n}{(2n+1)!} \qquad \text{roots are } (\pi n)^2 = r_n$$

$n \geq 1.$

$$\sum_{n=1}^{\infty} \frac{1}{r_n} = \sum_{n=1}^{\infty} \frac{1}{(\pi n)^2} = \frac{1}{\pi^2} \zeta(2).$$

$$g(t) = 1 - \frac{t}{6} + \frac{t^2}{120} - \frac{t^3}{720} \cdots$$

$$a_1 = -\frac{1}{6} \text{ “}\Rightarrow\text{” } \frac{1}{6} = \frac{\zeta(2)}{\pi^2} \Rightarrow \zeta(2) = \frac{\pi^2}{6}.$$

$$g(t) = 1 + a_1 t + a_2 t^2 + \cdots \qquad g(0) = 1$$

$$a_1 = -\sum \frac{1}{r_i}, \quad a_2 = +\sum_{i<j} \frac{1}{r_i r_j}$$

$$\sum_{i<j} \frac{1}{r_i r_j} = \frac{1}{2} \sum_{i \neq j} \frac{1}{r_i r_j} = \frac{1}{2} \sum_{i,j} \frac{1}{r_i r_j} - \frac{1}{2} \sum_i \frac{1}{r_i^2}.$$

$$= \frac{1}{2}\left(\sum_i \frac{1}{r_i}\right)^2 - \frac{1}{2} \sum_i \frac{1}{r_i^2}$$

Apply to ~~sine~~ $g(t) = \dfrac{\sin(\sqrt{t})}{\sqrt{t}}$.

$$= 1 - \frac{t}{6} + \frac{t^2}{120} - \frac{t^3}{720} \cdots$$

$$\frac{1}{r_i} = \frac{1}{\pi^2}\zeta(2), \quad \frac{1}{r_i^2} = \frac{1}{\pi^4} \sum_i \frac{1}{n^4} = \frac{1}{\pi^4}\zeta(4)$$

$$r_n = \frac{1}{(\pi n)^2} \qquad r_n^2 = \frac{1}{(\pi n)^4}$$

$$\frac{1}{2\pi^4}\left(\zeta(2)\right)^2 - \frac{1}{2\pi^4}\zeta(4) \text{ ``}=\text{'' } \frac{1}{120}.$$

$$\frac{1}{2\pi^4}\left(\frac{\pi^2}{6}\right)^2 - \frac{1}{2\pi^4}\zeta(4) = \frac{1}{120} \Rightarrow \zeta(4) = \frac{\pi^4}{90}.$$

$$-\frac{1}{720} = -\sum_{m<n<p} \frac{1}{(\pi m)^2(\pi n)^2(\pi p)^2}$$

$$x_1, x_2, \ldots, x_r, \ldots$$

$$\varepsilon_k(x_1, x_2, \ldots) = \sum_{i_1<\cdots<i_k} x_{i_1}\cdots x_{i_k}.$$

$$P_k(x_1, \dots, x_n, \dots) = \sum_{i=1}^{\infty} x_i^k$$

$$E_k(x_1, x_2, \dots) = \Psi_k(P_1, P_2, \dots, P_k) \quad Q$$

Terms coming in are $P_{i_1} \cdots P_{i_\ell} \cdot a_{i_1 \dots i_\ell}$

such that $i_1 + \dots + i_\ell = k$ only one

term $a_k P_k$ involves p_k. $a_k \neq 0$

$a_{1\dots 1} P_1^k + a_{1\dots 12} P_1^{k-2} P_2 \dots$ Newton showed

how to calculate.

$$\varepsilon_1 = p_1$$
$$\varepsilon_2 = \frac{1}{2} p_1^2 - \frac{1}{2} p_2$$
$$\varepsilon_3 = \frac{1}{6} p_1^3 - \frac{1}{2} p_1 p_2 + \frac{1}{3} p_3$$

<u>Ex.</u> Calculate $\zeta(6)$.

By induction we have:

$$\zeta(2n) = \pi^{2n} c_n \quad , \quad c_n \in \mathbb{Q}.$$

$$g(t) = \sum (-1)^n \frac{t^n}{(2n+1)!}, \quad g(0) = 1$$

Theorem.

$$g(t) = \prod_{n=1}^{\infty} \left(1 - \frac{t}{(\pi n)^2}\right).$$

$$h(x) = x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{(\pi n)^2}\right) = \sin(x)$$

$$\Longrightarrow \quad h(x+\pi) = -h(x).$$

$$(x+\pi)\left(1 - \frac{(x+\pi)^2}{\pi^2}\right)\left(1 - \frac{(x+\pi)^2}{4\pi^2}\right) \cdots$$

$$\frac{\pi^2-(x+\pi)^2}{\pi^2} = \frac{(\pi-(x+\pi))(\pi+(x+\pi))}{\pi^2}$$

$$= -x\left(\frac{x+2\pi}{\pi^2}\right).$$

$$\frac{4\pi^2-(x+\pi)^2}{4\pi^2} = \frac{(2\pi - x - \pi)(2\pi+x+\pi)}{4\pi^2}$$

$$\frac{(\pi-x)(3\pi+x)}{4\pi^2}$$

$$+x\cdot\left(1-\frac{x^2}{\pi^2}\right)\left(\frac{(2\pi+x)(3\pi+x)}{4\pi^2}\right)\cdots$$

$$-x\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{4\pi^2}\right)\frac{(3\pi+x)(4\pi+x)}{(6\pi)^2}\cdots$$

and in any conclude

$$f(x+\pi) = \pm f(x).$$

$$f(t,x) = \prod_i (1 - t x_i)$$

we leave the number of i's undetermined.

$$\frac{\partial f(t,x)}{\partial t} = \sum_j -x_j \cdot \prod_{i \neq j} (1 - t x_i)$$

$$= \sum_j \frac{-x_j}{(1-tx_j)} \cdot \prod_i (1-tx_i)$$

$$\frac{-x_j}{1-tx_j} = -x_j \sum_{k=0}^{\infty} (tx_j)^k = -\sum_{k=1}^{\infty} t^{k-1} x_j^k$$

$$-\sum_j \frac{x_j}{1-tx_j} = -\sum_{k=1}^{\infty} t^{k-1} p_k(x).$$

$$p_k(x) = \sum_{j\geq 1} x_j^k.$$

$$\frac{\partial f}{\partial t}(t,x) = -\sum_{k\geq 1} t^{k-1} p_k(x) f(t,x).$$

$$f(t,x) = \prod(1 - t x_i) = \sum (-1)^j t^j \varepsilon_j(x_1, x_2, \dots)$$

$$\varepsilon_1(x) = \sum x_i \qquad \varepsilon_3(x) = \sum_{i_1<i_2<i_3} x_{i_1} x_{i_2} x_{i_3} \dots$$

$$\varepsilon_2(x) = \sum_{i<j} x_i ,$$

$$\frac{\partial f}{\partial t}(t,x) = -\sum_{k\geq 1} t^{k-1} p_k(x) \cdot \sum_j (-1)^j t^j \varepsilon_j(x)$$

$$= \sum_{r\geq 1} t^{r-1} \sum_{l=0}^{r-1} (-1)^{l+1} p_{r-l}(x)\, \varepsilon_l(x).$$

$$= \sum_{r\geq 1} (-1)^r r\, t^{r-1} \varepsilon_r(x)$$

We have Newton's formula:

$$(-1)^r r \cdot \varepsilon_r(x) = \sum_{l=0}^{r-1} (-1)^{l+1} P_{r-l}(x)\, \varepsilon_l(x).$$

$r=1$

$$-\varepsilon_1(x) = -P_1(x)\,\varepsilon_0(x), \qquad \varepsilon_0(x) = 1$$

$$\varepsilon_1(x) = P_1(x)$$

$r=2$

$$2\varepsilon_2(x) = -P_2(x) + P_1(x)^2$$

$$\Rightarrow \quad \varepsilon_2(x) = \frac{1}{2}\left(P_1(x)^2 - P_2(x)\right)$$

$$-3\varepsilon_3(x) = -P_3(x) + P_2(x)P_1(x) - P_1(x)\varepsilon_2(x)$$

$$= -P_3(x) + P_2(x)P_1(x) - P_1(x)\cdot\left(\frac{1}{2}P_1(x)^2 - \frac{1}{2}P_2(x)\right)$$

$$= -P_3(x) + \frac{3}{2}P_2(x)P_1(x) - \frac{1}{2}P_1(x)^3$$

$$\varepsilon_3(x) = \frac{1}{3!}\left(2P_3(x) - 3P_2(x)P_1(x) + P_1^3\right)$$

We prove by induction

$$\varepsilon_k(x) = \frac{1}{k!}\sum_{\substack{i_1+\dots+i_r=k \\ 1\le i_1\le\dots\le i_r}} a_{i_1\dots i_r}\, P_{i_1}P_{i_2}\cdots P_{i_r}$$

and $a_{i_1\dots i_r} \in \mathbb{Z}$.

$$(-1)^{k+1}(k+1)\,\varepsilon_{k+1} = \sum_{l=0}^{k} (-1)^{l+k+1} p_{k+1-l}\,\varepsilon_l$$

$$= \sum_{l=0}^{k} (-1)^{l+k+1} p_{k+1-l} \frac{1}{l!} \sum_{\substack{i_1+\cdots+i_r=l \\ 1\le i_1\le\cdots\le i_r}} a_{i_1\cdots i_r} p_{i_1}\cdots p_{i_r}$$

$$= \frac{1}{k+1}\sum_{l=0}^{k} (-1)^{k+l} p_{k+1-l} \frac{1}{l!} \sum a_{i_1\cdots i_r} p_{i_1}\cdots p_{i_r}$$

$$= \frac{1}{(k+1)!}\sum_{l=0}^{k} (-1)^{k+l} p_{k+1-l} \frac{k!}{l!} \sum_{i_1+\cdots+i_r=l,\ 1\le i_1\le\cdots\le i_r} a_{i_1\cdots i_r} p_{i_1}\cdots p_{i_r}$$

QED.

$$\prod_{n=1}^{\infty}\left(1-\frac{t}{(\pi n)^2}\right) = \sum_{n=0}^{\infty}(-1)^n \frac{t^n}{(2n+1)!}$$

$\uparrow \pi$

$$\Rightarrow (-1)^n \frac{1}{(2n+1)!} = \varepsilon_n\left(\frac{1}{\pi^2 \cdot 1^2}, \frac{1}{\pi^2 \cdot 2^2}, \frac{1}{\pi^2 \cdot 3^2}, \cdots\right)$$

$$= \frac{1}{\pi^{2n}}\varepsilon\underbrace{\left(\frac{1}{1^2}, \frac{1}{2^2}, \frac{1}{3^2}, \cdots\right)}$$

$$= \frac{1}{\pi^{2n} \cdot n!}\left(\sum_{\substack{i_1 \le \cdots \le i_r \\ i_1 + \cdots + i_r = n}}' a_{i_1 \cdots i_r} P_{i_1}(\text{---}) P_{i_2}(\cdots) \cdots P_{i_r}(\text{---})\right)$$

$$P_j\left(\frac{1}{1^2}, \frac{1}{2^2}, \dots\right) = \zeta(2j)$$

Theorem (Euler) $$(-1)^n \frac{1}{(2n+1)!} = \frac{1}{\pi^{2n}\, n!} \sum_{\substack{1 \le i_1 \le \cdots \le i_r \\ i_1 + \cdots + i_r = n}} a_{i_1 \cdots i_r}\, \zeta(2i_1)\zeta(2i_2)\cdots\zeta(2i_r)$$

Cor. $\zeta(2j) = \pi^{2j} c_j \qquad c_j \in \mathbb{Q}$.

Proof. By induction. **Exercise.**

Ex. Calculate $a_{i_1 \cdots i_r}$ where $1 \le i_1 \le \cdots \le i_r = 4$.

Partition of n:

$i_1 \geq \cdots \geq i_r \geq 1 \quad i_1 + \cdots + i_r = n.$

Example. ℓ_8

$n=1$	1	1
$n=2$	2	2
	1 1	
$n=3$	3	
	2 1	3
	1 1 1	
$n=4$	4	
	3 1	
	2 1 1	5
	2 2	
	1 1 1 1	

Exercise: Pari program $p(n)$

number of part. of n.

$$\mathrm{mypart}(m,n) = \mathrm{if}\big((m>n), 0, \mathrm{if}(m==n, 1, \mathrm{mypart}(m, n-m) + \mathrm{mypart}(m+1, n))\big)$$

Ex: Prove $\mathrm{mypart}(1,n) = p(n)$.

Calculate $p(5)$, $p(6)$, $p(20)$, $p(100)$.