Solutions to Practice Midterm (Fall 2007)

November 2, 2007

1. The contrapositive of "If n is divisible by 4 then n is even" is "If n is odd then n is not divisible by 4."

2.

P	Q	not(P and Q)	not(P or (not Q))	(not P) or (not Q)	not ($P \Longrightarrow Q$)
T	T	F	F	F	F
T	F	T	F	T	T
F	T	T	T	T	F
F	F	T	F	T	F

We see therefore that the first and third statements are equivalent.

3. **(a)** By axiom (iii) there exists $c \in A$ such that $b + c = 0$. (We label each equality with the axiom that justifies it...)

$$a \overset{(i)}{=} a + 0 = a + (b + c) \overset{(iv)}{=} (a + b) + c = b + c = 0$$

(b) By axiom (iii) there exists $d \in A$ such that $a + d = 0$. Then

$$\begin{aligned} b &\overset{(i)}{=} b + 0 \\ &\overset{(ii)}{=} 0 + b \\ &= (a + d) + b \\ &\overset{(ii)}{=} (d + a) + b \\ &\overset{(iv)}{=} d + (a + b) \\ &= d + (a + c) \\ &\overset{(iv)}{=} (d + a) + c \\ &\overset{(ii)}{=} (a + d) + c \\ &= 0 + c \\ &\overset{(i)}{=} c \end{aligned}$$

4. **Claim**: $\sum_{m=1}^{n} m^2 = \frac{n(n+1)(2n+1)}{6}$

Proof: We prove the claim by induction on n.

base case: Since

$$\sum_{m=1}^{1} m^2 = 1^2 = 1 = \frac{1(1+1)(2(1)+1)}{6}$$

the claim is true for $n = 1$.

inductive hypothesis: Let $k \geq 1$ and assume that the claim holds for $n = k$. In other words we get to assume that

$$\sum_{m=1}^{k} m^2 = \frac{k(k+1)(2k+1)}{6}$$

inductive step: Now we must show that the claim holds for $n = k + 1$. Indeed

$$\begin{aligned}
\sum_{m=1}^{k+1} m^2 &= \sum_{m=1}^{k} m^2 + (k+1)^2 \text{ (by definition of the sum)} \\
&= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \text{ (by the inductive hypothesis)} \\
&= \frac{k(k+1)(2k+1)}{6} + \frac{6(k+1)^2}{6} \\
&= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} \\
&= \frac{(k+1)(k(2k+1) + 6(k+1))}{6} \\
&= \frac{(k+1)(2k^2 + 7k + 6)}{6} \\
&= \frac{(k+1)(k+2)(2k+3)}{6}
\end{aligned}$$

Therefore we have shown that

$$\sum_{m=1}^{k+1} m^2 = \frac{(k+1)(k+2)(2k+3)}{6}$$

and so the claim holds for $n = k + 1$.

By induction the proof is complete.

5. **(a)** **Claim**: For $n \geq 2$ $A_n = F_n + 2F_{n-1}$

Proof: We prove this by (strong) induction on n. This means that for the inductive hypothesis we want to assume that our claim is true for $n = k$ and $n = k - 1$. To be able to do this we have to prove two base cases.

base case: Here the base case is for the cases $n = 2$ and $n = 3$:

$$\begin{aligned} A_2 &= 3 \\ F_2 + 2F_1 &= 3 \end{aligned}$$

$$\begin{aligned} A_3 &= A_1 + A_2 = 4 \\ F_3 + 2F_2 &= 2 + 2(1) = 4 \end{aligned}$$

So the claim holds for $n = 2, 3$.

inductive hypothesis: Let $k \geq 3$ and assume that the claim holds for $n = k$ and $n = k - 1$. In other words we assume that

$$\begin{aligned} A_k &= F_k + 2F_{k-1} \\ A_{k-1} &= F_{k-1} + 2F_{k-2} \end{aligned}$$

inductive step: Now we have to show that the claim holds for $n = k + 1$. Indeed

$$\begin{aligned} A_{k+1} &= A_k + A_{k-1} \text{ (by definition of } A_{k+1}) \\ &= (F_k + 2F_{k-1}) + (F_{k-1} + 2F_{k-2}) \text{ (by the inductive hypothesis)} \\ &= F_{k+1} + 2(F_k) \text{ (by definition of } F_{k+1}, F_k) \end{aligned}$$

Therefore the claim holds for $n = k + 1$.

By induction the proof is complete.

(b) To discover the relationship between the B_n and F_n we should write down the first few term of both series and compare them. Then we can get a conjecture to try to prove.

n	B_n	F_n
1	1	1
2	m	1
3	$m+1$	2
4	$2m+1$	3
5	$3m+2$	5
6	$5m+3$	8
7	$8m+5$	13

Can you see the pattern? The coefficients of m form a Fibonacci sequence and the constant terms starting with $n = 3$ form their own Fibonacci sequence. So we conjecture that $B_n = F_{n-1}m + F_{n-2}$. Notice that we can rewrite this as $B_n = F_n + (m-1)F_{n-1}$. Now this looks analogous to part (a). Let's prove this:

Claim: For $n \geq 2$ $B_n = F_n + (m-1)F_{n-1}$

Proof: Again, we prove this by strong induction but we won't be as wordy this time.

base case: We prove the claim for the cases $n = 2, 3$:

$$\begin{aligned} B_2 &= m \\ F_2 + (m-1)F_1 &= 1 + (m-1)1 = m \end{aligned}$$

$$\begin{aligned} B_3 &= m+1 \\ F_3 + (m-1)F_2 &= 2 + (m-1)1 = m+1 \end{aligned}$$

So the base case holds.

inductive hypothesis: Let $k \geq 3$. We assume that

$$\begin{aligned} B_{k-1} &= F_{k-1} + (m-1)F_{k-2} \\ B_k &= F_k + (m-1)F_{k-1} \end{aligned}$$

inductive step: We have to prove that the claim is true for $n = k+1$. Indeed

$$\begin{aligned} B_{k+1} &= B_k + B_{k-1} \\ &= F_k + (m-1)F_{k-1} + F_{k-1} + (m-1)F_{k-2} \\ &= F_{k+1} + (m-1)F_k \end{aligned}$$

Therefore the case $n = k+1$ holds.

By induction the proof is complete.

6. **a**. The function is neither injective (since for instance $f(1) = f(-1)$) nor surjective (since the strictly negative real numbers are not in the image of f).

b. The function is injective since for two nonnegative real numbers x and y, if $x^4 = y^4$ then $x = y$. But it's not surjective (see part a).

c. The function is injective and surjective. To see that f is injective notice that if $x, y \in \mathbb{R}^+$ then

$$\frac{1}{1+x} = \frac{1}{1+y} \Longrightarrow 1+x = 1+y \Longrightarrow x = y$$

To see that f is surjective let $y \in \{x \in \mathbb{R} : 0 < x < 1\}$. Then let $x = \frac{1-y}{y}$. Notice that $x \in \mathbb{R}^+$ and $f(x) = y$.