Practice Final Exam Math 109

1. A mathematical statement that is always true is called a tautology. Which of the following statements are tautologies? You must justify your answers. (hint: use truth tables):

a) $((not\ P) \text{ or } Q) \Longrightarrow (P \Longrightarrow Q)$.
b) $(P \text{ or } Q)$ and (not Q).
c) $(P \Longrightarrow Q)$ or $(Q \Longrightarrow P)$.

2. Recall that the Fibonacci sequence is defined by $u_1 = 1, u_2 = 1$ and assuming that u_{n-1} and u_n have been defined then $u_{n+1} = u_n + u_{n-1}$. Prove that $u_n \geq (\frac{3}{2})^n$ for all $n \geq 11$. You may assume that $(3/2)^{11}$ has decimal expansion 86.4976.... (Hint: Prove by induction, you will get partial credit if you set up the induction correctly explaining what must be done in each step.)

3. Write the addition and multiplication tables for $\mathbb{Z}_8$. Find all solutions to $x^2 = 1$ in $\mathbb{Z}_8$. If $x \in \mathbb{Z}_8$ is invertible what is its inverse?

4. Fix $m > 1$. Prove that if $x \equiv m - 1 \bmod m$ or $x \equiv 1 \bmod m$ then $x^2 \equiv 1 \bmod m$. Show that if m is prime this describes all congruence classes modulo m of solutions to $x^2 \equiv 1 \bmod m$. If m is not prime is this assertion true?

5. One of the following statements is true for all $n \in \mathbb{Z}^+ = \{1, 2, 3, ...\}$. Decide which is true and prove that it is tre true and that the others are false.

a) $1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n} = \frac{(6n^2-13n+11)}{4}$.
b) $1 + \frac{1}{2} + \frac{1}{4} + ... + \frac{1}{2^n} = 2 - \frac{1}{2^n}$.
c) $(1 - \frac{1}{n+1})^n \geq \frac{1}{2}$.

6. Prove that the set of all integers, n, such that $36n^2 \geq n^4$ is finite. What is the cardinality of this set? (You must prove your assertion.)

7. Consider the function $f : \{x | 0 < x < 1\} \to \mathbb{R}^+ = \{x | x > 0\}$, defined by $f(x) = \frac{x}{1-x}$. Determine if it is injective (one-to-one). Is it surjective (onto)?

8. Find all solutions to the linear congruence equation $54x \equiv 36 \bmod 63$.

9. Prove that 223 divides $2^{37} - 1$ using modular arithmetic. (Hint: Show that $2^8 \equiv 33 \bmod 223$ so $2^{16} \equiv 33^2 \bmod 223$ and $33^2 \equiv 197 \bmod 223$. Use this to prove that $2^{32} \equiv 7 \bmod 223$.)